Pricing Options
in Markov-Modulated Fractional Brownian Markets

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Abstract

The Markov-modulated $(B, S)$-securities market is a $(B, S)$-security market, consisting of riskless asset, bond $B$, and risky asset, stock $S$, in random media $X$, or $(B, S)$-security market driven by a Markov process $x_t \in X$. We study the pricing options for Markov-modulated fractional Brownian $(B, S)$-security markets, including Hu & Øksendal (1999) and Elliott & van der Hoek (2000) schemes.

Incompleteness of Markov-modulated fractional Brownian $(B, S)$-security markets inHu & Øksendal and Elliott & van der Hoek schemes without and with jumps are established and Black-Scholes formulae for these schemes are derived.

Perfect hedging in a Markov-modulated Brownian fractional $(B, S)$-security market (without and with jumps) is not possible since we have an incomplete market. Following the idea proposed by Föllmer and Sondermann (1986) and Föllmer and Schweizer (1993) we look for the strategy locally minimizing the risk. The residual risk processes are presented in these two schemes.
1 Introduction

Consider a standard probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) with filtration \(\mathcal{F}_t\) and probability \(P\).

A Brownian \((B, S)\)-security market will consist of a riskless asset (bond or bank account) \(B = \{B_t, t \geq 0\}\) and risky asset, (stock or share) \(S = \{S_t, t \geq 0\}\) which satisfy the following system of two equations:

\[
\begin{align*}
\frac{dB_t}{B_t} &= r_t B_t, & \text{for } B_0 > 0, r > 0, \\
\frac{dS_t}{S_t} &= S_t (\mu dt + \sigma dB_t), & \text{for } S_0 > 0, \sigma > 0, \mu \in \mathbb{R}.
\end{align*}
\]

Here \(r\) denotes interest rate, \(\mu\) an appreciation rate, \(\sigma\) a volatility and \(w = \{w_t, t \geq 0\}\) is an \(\mathcal{F}_t\)-Brownian motion on \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\), \(Ew_t = 0\), \(Ew_t^2 = t\), where \(E\) is an expectation with respect to the measure \(P\). The second stochastic differential equation (SDE) involves an \(Itô\) integral. It is well-known that this market has no arbitrage and is complete (see Elliott and Kopp (1999)).

However, even in the Brownian motion framework, there is an arbitrage opportunity if Stratonovich or pathwise integration is used in the definition of stochastic integral (see Rogers (1997), Shiryae (1998)).

It is well-known that stock logreturns, i.e. \(\log \frac{S_t}{S_{t-1}}\), where \(S_t\) is the stock price at time \(t\), have fat-tailed distributions. Due to this fact many models for security markets have been suggested: Levy processes, jump-diffusions, processes with stochastic volatilities, subordinated processes and processes driven by fractional Brownian motion (fBm). All the processes, except fBm, generate incomplete markets.

A fractional Brownian \((B, S)\)-security market (fBm for short) consists of a riskless asset, (a bond or bank account) \(B\) and a risky asset, (a stock or share), \(S\) which satisfy the following system of two equations:

\[
\begin{align*}
\frac{dB_t}{B_t} &= r_t B_t, & \text{for } B_0 > 0, r > 0, \\
\frac{dS_t}{S_t} &= S_t (\mu dt + \sigma dB_t^H), & \text{for } S_0 > 0, \sigma > 0, \mu \in \mathbb{R}.
\end{align*}
\]

Here \(r\) is a constant interest rate, \(\mu\) is an appreciation rate, \(\sigma\) is a volatility and \(B_t^H\) is a fractional Brownian motion with Hurst index \(H \in (0, 1)\). This is a Gaussian process \(B = \{B_t^H, t \geq 0\}\) with mean \(EB_t^H = 0\) and covariance

\[
E[B_s^H B_t^H] = \frac{1}{2} \left[(s^2 - s^H) \wedge (t^2 - t^H)ight],
\]

for all \(s, t > 0\). Here \(E\) denotes the expectation with respect to the probability \(P\), \(B_0^H = 0\). If \(H = 1/2\), then \(B^H\) coincides with standard Brownian motion \(w\). If \(H < 1/2\), increments of the process are negatively correlated, if \(H > 1/2\) they are positively correlated. For any \(H \in (0, 1)\) the process \(B_t^H\) is self-similar in the sense that \(B_{\alpha t}^H\) has the same law as \(\alpha B_t^H\) for any \(\alpha > 0\).

In the second equation (2) the Wick product is used to define the integral \(\int_0^t S_u dB_u^H\), (see Elliott and van der Hoek (2000)). Duncan et al. (2000) introduced the Wick product in the definition of the stochastic integral for fBm.

In case the Wick product is used in the definition of stochastic integral in (2), Hu and Øksendal (1999) and Elliott and van der Hoek (2000) show that there is no arbitrage.
Because of these properties, $B^H$ with Hurst parameter $H \in (1/2, 1)$ has been suggested as a model in applications, including finance (see Mandelbrot (1997)). Cutland et al. (1993) proposed the long range dependence for stock price dynamics. Corazza et al. (1997) empirically studied foreign currency markets which supports the multi-fractal market hypothesis. Empirical evidence that a Hurst parameter with values in $(1/2, 1)$ is appropriate for foreign exchange rates was provided by Los et al. (1997).

However, it was discovered that mathematical markets based on $B^H$ could have arbitrage. This is the case even for the fractional analogue of the basic Black-Scholes market. However, the first $B^H$-integration theory was based on using ordinary products and led to on an integral which we shall denote by

$$\int_a^b f(t, \omega) \delta B^H_t.$$

These integrals do not have expectation zero and we shall call them fractional Stratonovich integrals, by analogy with the situation for standard Brownian motion. For this model of a fractional Black-Scholes market we again have arbitrage, Rogers (1997).

**Remark 1.** Note that if $S_t = e^{rt + \alpha w_t + \sigma B^H_t}$

then the arbitrage in the above sense is impossible (see Mishura & Valkeila (2000)), where $w$ is a standard Wiener process.

**Remark 2.** If we can use a fractional Itô integral with respect to $B^H_t$, then the corresponding Itô fractional Black-Scholes market has no arbitrage, when $H \in (1/2, 1)$. Moreover, this fractional Black-Scholes market is complete (see Hu & Øksendal (1999)).

**Remark 3.** The fractional Black-Scholes market, developed by Elliott & van der Hoek (2000) for all Hurst indices $H \in (0, 1)$, using one probability measure $P$, is also arbitrage free. Furthermore, this market is complete.

By a Markov-modulated fractional Brownian $(B, S)$-security market we mean a securities market which consists of riskless asset (bond or bank account) $B_t$ and risky asset (stock or share) $S_t$ that satisfy the following system of two equations:

$$\begin{cases} \quad dB_t = r(x_t) B_t, \quad B_0 > 0, \quad r(x) > 0, \\ \quad dS_t = S_t(\mu(x_t) dt + \sigma(x_t) dB^H_t), \quad S_0 > 0, \quad \sigma(x) > 0, \quad \mu(x) \in R. \end{cases}$$

(4)

Here $r(x)$ (an interest rate), $\mu(x)$ (an appreciation rate), $\sigma(x)$ (a volatility) are bounded continuous functions on $X$, the phase space of the Markov process $x_t$, and $B^H_t$ is a fractional Brownian motion with Hurst index $H \in (0, 1)$ independent of $x_t$. The main goal of this paper is to study the model (4), including such models with jumps.

**Literature Review.** It is well-known that fBm has many applications not only in finance (see Mandelbrot (1997)), but also in economics (time series exhibit cycles of all orders of magnitude), in fluctuations in solids, in hydrology, etc. (see Mandelbrot & van Ness (1968)). Hurst (1965) was the first one who found the range of cumulative water flows in hydrology to vary proportionally to $t^H$ with $1/2 < H < 1$. Mandelbrot & Van Ness (1968) considered such filtrations of all fBm that coincide with the filtration generated by the driving Brownian motion, Lindstrom (1993) obtained a representation of fBm which is not adapted to the filtration generated by the driving Brownian motion,
Cutland, Kopp & Willinger (1995) proposed the long range dependence for stock price dynamics. Lin (1995) developed integration theory for fBm on the ordinary pathwise product (Fisk-Stratonovich integral). Corazza & Malliaris (1997) empirically studied foreign currency markets which support the multi-fractal hypothesis. Rogers (1997) remarked that arbitrage occurs if we use the Fisk-Stratonovich integral in the definition of self-financing portfolios for fractional Brownian market. Shiryaei (1998) mentioned that arbitrage occurs if we use the Fisk-Stratonovich integral in the definition of self-financing portfolios for a Brownian market. Decreusefond & Ustunel (1998) proposed to use Malliavin calculus to define the integral wrt to fBm. Hu & Øksendal (1999) obtained an option pricing formulae for fractional Brownian markets with Hurst index $H \in (1/2, 1)$ (no arbitrage here, market is complete). Duncan, Hu & Pasik-Duncan (2000) introduced the Wick product in the definition of stochastic integral for fBm. Alós, Mazet & Nualart (2000, 2001) proposed an equivalent approach to the definition of integral wrt a fBm as divergence operator via the Malliavin calculus. Stochastic stability of fractional $(B, S)$-security markets (in both Hu & Øksendal and Elliott & van der Hoek schemes) was considered in Mishura and Swishchuk (2001). Brody et al. (2002) proposed to use an Ornstein-Uhlenbeck process driven by a fBm to model weather derivatives. Elliott & van der Hoek (2003) obtained option pricing formulae for fractional Brownian market with Hurst index $H \in (0, 1)$ (no arbitrage here, market is complete). Bender (2003) constructed fBm for all Hurst parameter $0 < H < 1$ on the same probability space and proved an Itô formula for generalized functionals of fBm. Björk & Hult (2005) noted that fractional Black-Scholes models admit arbitrage (problem contained in the definitions of the self-financing trading strategies and of the value of a portfolio). Elliott & Swishchuk (2004) studied options pricing formula and swaps for Markov-modulated Brownian and fractional Brownian Markets with jumps. The jumps in the dynamic of stock prices have been included, in particular, by Merton (1973) and Aase (1988).

The paper is organized as follows.

In Sections 2 and 3 we consider incompleteness of Markov-modulated fractional Brownian $(B, S)$-security markets (4) in Hu & Øksendal and Elliott & van der Hoek schemes schemes, respectively, without (Section 2.2) and with jumps (Section 2.3). Black-Scholes formulae for these schemes are derived in Sections 2.4-2.5 and Sections 3.4-3.5, respectively.

Perfect hedging in a Markov-modulated Brownian fractional $(B, S)$-security market (without and with jumps) is not possible since we have an incomplete market. Following the idea proposed by Föllmer & Sondermann (1986) and Föllmer & Schweizer (1993) we look for the strategy locally minimizing the risk. The residual risk processes are presented for these two schemes.
2 Pricing Options for Markov-Modulated Fractional Brownian \((B, S)\)-Security Markets (Hu & Øksendal Scheme)

2.1 Fractional \((B, S)\)-Security Market (Hu & Øksendal Scheme (1999))

We note that the fractional \((B, S)\)-securities market defined by Hu & Øksendal (1999) has two investment possibilities:

1) A bank account or a bond, where the price \(B_t\) at time \(t\) develops according to the equation

\[
dB_t = r B_t dt, \quad B_0 > 0, \quad r > 0;
\]

2) A stock, where the price \(S_t\) at time \(t\) satisfies the equation

\[
dS_t = \mu S_t dt + \sigma S_t dB^H_t, \quad S_0 > 0, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad H \in (1/2, 1), \quad t \in [0, T].
\]

Here \(B^H_t\) is a fractional Brownian motion with zero mean and \(E(B^H_t)^2 = t^{2H}\), \(H \in (1/2, 1)\). From Hu & Øksendal (1999) we know that

\[
S_t = S_0 e^{\sigma B^H_t + \mu t - \frac{1}{2} \sigma^2 t^{2H}}, \quad t \geq 0.
\]

The integral in (38) is the Itô integral. As was stated in Hu & Øksendal (1999), this fractional \((B, S)\)-securities market is complete.

2.2 Markov-Modulated Fractional Brownian \((B, S)\)-Security Markets in the Hu & Øksendal Scheme

Let us consider the fractional \((B, S)\)-security market

\[
\begin{cases}
    dB_t = r(x_t) B_t dt, & B_0 > 0, \\
    dS_t = S_t (\mu(x_t) dt + \sigma(x_t) dB^H_t), & S_0 > 0,
\end{cases}
\]

where \(x_t\) is a homogeneous Markov process independent of \(B^H_t\) with infinitesimal operator \(Q\), \(\mu(x)\) and \(\sigma(x)\) are continuous and bounded functions on the phase space of states \(X\).

We note that the system (40) can be rewritten in another form

\[
\frac{dS_t}{dt} = \mu(x_t) S_t + \sigma(x_t) S_t \triangle W^H_t,
\]

where \(W^H_t\) is the fractional white noise, \(\triangle\) is a Wick product (see Hu & Øksendal (1999)), and

\[
\frac{dB^H_t}{dt} = W^H_t \in (S)^*_H,
\]
where \((S)^\ast(R) =: \Omega\) (i.e. \(\Omega\) is the space of tempered distributions \(\omega\) on \(R\)) is a dual space to \(S(R)\) (the Schwartz space of rapidly decreasing smooth functions on \(R\)) (see Hu & Øksendal (1999)). We note that
\[
S =: \exp(\mathcal{E}) = \{ f : R \to \Omega : \int_R \int_R f(s)f(t)\phi(s,t)dsdt < + \infty \}.
\]
Hence, from (8)-(9) we have
\[
\frac{dS_t}{dt} = (\mu(x_t) + \sigma(x_t)W_t^H) \bowtie S_t.
\] (11)
Using the Wick product we see that the solution of this equation is
\[
S_t = S_0 \exp^\phi\left(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)W_s^Hds\right) = S_0 \exp^\phi\left(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)dB_s^H\right).\] (12)
We note that
\[
\exp^\phi(<\omega, f>) = \exp(<\omega, f> - \frac{1}{2}[f^2_\phi]) = \mathcal{E}(f) := \exp\left(\int_R fdB^H - \frac{1}{2}[f^2_\phi]\right)
\] (13)
for all \(f \in L^2_\phi(R)\). Hence, from (12)-(13) we obtain
\[
S_t = S_0 \exp^\phi(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)W_s^Hds) = S_0 \exp^\phi(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)dB_s^H) = S_0 \exp(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)dB_s^H) - \frac{1}{2} \int_0^t \int_0^t \sigma(x_s)\sigma(x_u)\phi(s,u)dsdu,
\] (14)
where \(\phi(s, u)\) is defined in (10).
Let \(K^H_H(s, x)\) be a function such that
\[
\int_R K^H_H(s, x)K^H_H(t, x)\phi(s, t)dsdt = \gamma(x),
\] (15)
where
\[
\gamma(x) := (r(x) - \mu(x))/\sigma(x).
\] (16)
Define the following measure \(P^H_\phi:\)
\[
\frac{dP^H_\phi}{d\mu_\phi} = \exp\left(\int_0^T K^H_H(s, x_s)dB^H_s - \frac{1}{2} \int_0^T \int_0^T K^H_H(s, x_s)K^H_H(u, x_u)\phi(s, u)dsdu\right) := \eta^H_T,
\] (17)
where the measure $\mu_\phi(\omega)$ on $\Omega$ is defined by the Bochner-Minlos theorem as follows:

$$
\int_{\Omega} e^{i\langle \omega, f \rangle} d\mu_\phi(\omega) = \exp\left(-\frac{1}{2}|f|^2_2\right),
$$

for all $f \in S(R)$-Schwartz space of rapidly decreasing smooth functions on $R$, and $|f|^2_2 := \int f(s)\phi(s,u)dsdu < +\infty$.

We note that $P^H_\phi(\omega)$ is a probability measure.

By Girsanov theorem for fractional Brownian motion (Theorem 3.18, Hu & Øksendal (1999)) we see that

$$
\hat{B}^H_t := \int^t_0 \frac{\mu(x_s) - r(x_s)}{\sigma(x_s)} ds + B^H_t = B^H_t - \int^t_0 \gamma(x_s) ds
$$

(18)

is a fractional Brownian motion w.r.t. the measure $P^H_\phi$ defined in (17).

In terms of $\hat{B}^H_t$ the second equation in (7) may be written in the following form

$$
\frac{dS_t}{S_t} = r(x_t) dt + \sigma(x_t)d\hat{B}^H_t.
$$

(19)

Let

$$
Y_t := \frac{S_t}{\hat{B}^H_t},
$$

(20)

where $S_t$ and $B_t$ are defined in (7) and (19) respectively. Then $Y_t$ in (20) satisfies the following equation

$$
Y_t = Y_0 + \int^t_0 \frac{\sigma(x_u) S_u}{B_u} d\hat{B}^H_u.
$$

(21)

**Lemma 1.** The function $K^T_H(s,x)$ in (48) may be found explicitly by

$$
K^T_H(s,x) = \frac{\mu(x) - r(x)}{2\sigma(x)H(2H-1)T(2H-1)\Gamma(2-2H)\Gamma(H-1/2)}(T-s)^{1/2-H},
$$

(22)

where $\Gamma(s)$ is a Gamma function.

**Proof.** We note that the solution of the equation (15) is given as

$$
K^T_H(s,x) = -\frac{1}{H(2H-1)\Gamma_H} \frac{\gamma(x)}{s^{1/2-H}} \times \int^x_0 \left\{ \int^w_z \frac{1}{T} \frac{\Gamma(3/2-H)\Gamma(2-2H)}{\Gamma(2-2H)} dw \right\} dz,
$$

(23)

where $d_H := 2\Gamma^2(3/2-H)T(2H-1)\cos(\pi(H-1/2))$. It is well known [3] that

$$
\int^w_0 z^{1/2-H}(w-z)^{1/2-H} dz = \frac{T^2(3/2-H)}{T(3-2H)}w^{2-2H}.
$$

In this way,

$$
\int^w_0 z^{1/2-H}(w-z)^{1/2-H} dz = \frac{(2 - 2H)T^2(3/2 - H)}{T(3 - 2H)}w^{1-2H} = \frac{T^2(3/2 - H)}{\Gamma(2 - 2H)}w^{1-2H}.
$$

(24)
But
\[ \int_s^T (w - s)^{1/2-H} \, dw = (3/2 - H)^{-1} (T - s)^{3/2-H}. \]
Hence,
\[ \frac{d}{ds} \int_s^T w^{2H-1} (w - s)^{1/2-H} \, dw = - (T - s)^{1/2-H}. \]  
(25)

By (23)-(25) we have
\[ K^T_H(s, x) = K_H \frac{\mu(x) - r(x)}{\sigma(x)} s^{1/2-H} (T - s)^{1/2-H}, \]
where \( K_H := \frac{1}{2H(2H-1)(2H-1)\Gamma(2-2H)\cos(\pi H/2)} \) and Lemma 1 is proved.

2.3 Incompleteness of Markov-Modulated Fractional Brownian \((B, S)\)-Security Market in the Hu & Øksendal Scheme

Set \( \mathcal{F}^H_t \) be the \( \sigma \)-algebra generated by \( B^H_t \) and \( x_s \), \( 0 \leq s \leq t \).

**Definition 1.** A portfolio \( \pi_t := (\alpha_t, \beta_t) \) is an \( \mathcal{F}^H_t \)-adapted 2-dimensional process giving the number of units \( \alpha_t \) and \( \beta_t \) held at time \( t \) of the bond and the stock, respectively. The corresponding value process \( X^\pi_t \) is given by
\[ X^\pi_t = \alpha_t B_t + \beta_t \diamond S_t. \]  
(26)

**Definition 2.** The portfolio is called *admissible* if \( X^\pi_t \) is bounded below a.s. and self-financing, in the sense that
\[ dX^\pi_t = \alpha_t dB_t + \beta_t \diamond dS_t, \quad t \in [0,T]. \]  
(27)

Assume that portfolio \( (\alpha_t, \beta_t) \) is self-financing. By (26) we have
\[ \alpha_t = \frac{X^\pi_t - \beta_t \diamond S_t}{B_t}, \]  
(28)

which after substituting into (27) gives
\[ dX^\pi_t = r(x_t)X^\pi_t dt + \sigma(x_t)\beta_t \diamond S_t [\gamma_0 dt + dB^H_t]. \]  
(29)

In terms of \( \hat{B}^H_t \) in (51) the equation (62) may be written in the form
\[ dX^\pi_t = r(x_t)X^\pi_t dt + \sigma(x_t)\beta_t \diamond S_t d\hat{B}^H_t. \]  
(30)

Integrating (63) and multiplying by \( \exp\{- \int_0^t r(x_s) ds\} \) we get
\[ \exp\{- \int_0^t r(x_s) ds\} X^\pi_t = X^\pi_0 + \int_0^t \exp\{- \int_0^s r(x_u) du\} \sigma(x_s)\beta_s \diamond S_s d\hat{B}^H_s, \quad t \in [0,T], \]  
(31)

where \( X^\pi_0 \) is an initial value.
**Definition 3.** An admissible portfolio \( \pi \) is called an *arbitrage* for the market \((B_t, S_t)\), if \( X_0 \leq 0, \quad X_T > 0 \) a.s. and
\[
\mu_\phi \{ w : X_T^\pi(w) > 0 \} > 0.
\]

We note that under measure \( \hat{P}_\phi^H \) in (17)
\[
\begin{align*}
E_{\hat{P}_\phi^H} \hat{B}_t^H &= 0 \\
E_{\hat{P}_\phi^H} \int_0^t \exp \{- \int_0^s r(u) du \} \sigma(x_s) \beta_s \diamond S_s d\hat{B}_s^H &= 0.
\end{align*}
\] (32)

From (32), (31) and (17) we deduce that no arbitrage can exist, because by taking the expectation w.r.t. \( \hat{P}_\phi^H \) we get
\[
E_{\hat{P}_\phi^H} \exp \{- \int_0^T r(x_u) du \} X_T^\pi = X_0^\pi := X.
\]

**Definition 4.** The market \((B_t, S_t)\) is called *complete* if for every \( \mathcal{F}_t \)-measurable bounded r.v. \( F(\omega) \) there exists \( X \in R \) and portfolio \( \pi = (\alpha_t, \beta_t) \) such that
\[
F(\omega) = X_T^\pi(\omega) \quad \text{a.s.} \mu_\phi.
\]

By (31) this is the same as requiring that
\[
\exp \{- \int_0^T r(x_s) ds \} F(\omega) = X + \int_0^T \exp \{- \int_0^s r(x_u) du \} \sigma(x_s) \beta_s \diamond S_s d\hat{B}_s^H], \quad t \in [0, T].
\] (33)

If we apply the fractional Clark-Ocone theorem (Theorem 4.13, [3]) to
\[
G(\omega) := \exp \{- \int_0^T r(x_s) ds \} F(\omega)
\]
and with \( \hat{B}_t^H \) replaced by \( \tilde{B}_t^H \) we get
\[
\begin{align*}
\exp \{- \int_0^T r(x_s) ds \} F(\omega) &= E_{\hat{P}_\phi^H} \exp \{- \int_0^T r(x_u) du \} F(\omega) \\
&+ \int_0^T E_{\hat{P}_\phi^H} \exp \{- \int_0^T r(x_u) du \} \tilde{D}_t F/\mathcal{F}_t^H d\hat{B}_t^H.
\end{align*}
\] (34)

where \( \tilde{D}_t \) denotes the stochastic gradient with respect to \( \hat{P}_\phi^H \), and \( \tilde{E} \) is a quasi-conditional expectation. Note that the \( \sigma \)-algebra \( \mathcal{F}_t^H \) generated by \( \tilde{B}_t^H \) is the same as \( \mathcal{F}_t^H \).

Comparing (33) and (34) we obtain
\[
X = E_{\hat{P}_\phi^H} \exp \{- \int_0^T r(x_u) du \} F(\omega)
\]
and
\[
\exp \{- \int_0^T r(x_s) ds \} \sigma(x_s) \beta_s \diamond S_s = \tilde{E}_{\hat{P}_\phi^H} \exp \{- \int_0^T r(x_u) du \} \tilde{D}_t F/\mathcal{F}_t^H.
\]
There is a unique initial value (by construction)

\[ X = X_0^\pi = E_{\tilde{P}_H}^\pi \{ \exp\{ - \int_0^T r(x_u) du \} F(\omega) \}, \]

and a unique portfolio \( \pi_t = (\alpha_t, \beta_t) \) which we need to replicate (hedge) the claim \( F(\omega) \).

Introduce the following measure:

\[ \frac{d\hat{P}_H}{d\mu_\phi} = \eta_H E_T, \]

where \( \eta_H \) and \( E_T^\pi \) are defined in (17) and (13), respectively.

We note, that the measure \( \tilde{P}_H \) is a probability measure.

Using the same reasonings as in (31)-(34), we obtain that

\[ X = X_0^\pi = E_{\tilde{P}_H}^\pi \{ \exp\{ - \int_0^T r(x_u) du \} F(\omega) \}. \]

Thus, we can conclude that the two measures \( \tilde{P}_H \) and \( \hat{P}_H \) are equivalent. Since we have two measures, the market is incomplete. Hence, we can use measure \( \hat{P}_H \) to calculate the price of the claim \( F(\omega) \).

The initial value \( X \) is called the price of the claim \( F(\omega) \).

Summarizing all the above reasonings we obtain the following result.

**Theorem 1.** The Markov-modulated fractional \((B, S)\)-security market has no arbitrage. It is incomplete, and the price \( x \) of a lower bounded \( \mathcal{F}_t^H \)-measurable claim \( F(\omega) \in L^2(\hat{P}_H) \) is given by

\[ X = X_0^\pi = E_{\tilde{P}_H}^\pi \{ \exp\{ - \int_0^T r(x_u) du \} F(\omega) \}, \]

where \( \tilde{P}_H \) is defined in (17). The corresponding replicating/hedging portfolio \( \pi_t = (\alpha_t, \beta_t) \) for the claim \( F \) is

\[ \begin{align*}
\beta_t &= \exp\{ \int_0^t r(x_s) ds \} \sigma^{-1}(x_t) S_{t-}(1) \hat{D}_{t} F/\mathcal{F}_t^H, \\
\alpha_t &= \frac{N_t - \hat{H}_t \hat{S}_t}{B_t}. 
\end{align*} \]

2.4 The Black-Scholes Formula for Markov-Modulated Fractional Brownian \((B, S)\)-Security Market in the Hu & Øksendal Scheme

A claim of special interest is the European call, where

\[ F(\omega) := (S_T(\omega) - K)^+, \]

where \( K > 0 \) is the exercise price.

Using the following representation (see (14))

\[ S_T = S_0 e^{\int_0^T \sigma(x_u) du} B_T + \int_0^T \mu(x_u) du - \frac{1}{2} \int_0^T \sigma(x_u)^2 \sigma(x_u, \phi(s, u)) du du, \]
and the Black-Scholes formula for Markov-modulated \((B, S)\)-security markets (see Section 3.2) (see also (24)-(27)), we obtain the following result.

**Theorem 2.** Let \(r(x) \equiv r\) for all \(x \in \mathcal{X}\). The price \(C_0^H\) of the fractional European call is

\[
C_0^H = e^{-rT} \int_{\mathcal{F}} \frac{1}{\pi^2} \left( S_0 \exp \left\{ zy + rT - \frac{1}{2} z^2 T^{2H} \right\} - K \right) \frac{1}{\pi} \exp \left\{ - \frac{y^2}{2 \pi^2} \right\} \, dy F_T^z(dz)
\]

where

\[
F_T^z(dz) := P \left( \int_0^t \int_0^t \sigma(x_u) \sigma(x_v) \phi(s, u) ds du \in (z, z + dz) \right),
\]

\[
C_{BS}^H(\tilde{\sigma}, T) := S_0 \Phi(\tilde{d}_z^H) - Ke^{-rT} \Phi(\tilde{d}_z^H),
\]

and

\[
d_z^H = \left[ \log \frac{S_0}{K} + rT \pm \tilde{\sigma}^2 T^{2H} / \tilde{\sigma} T^H \right].
\]

**Remark 1.** Perfect hedging in a Markov-modulated Brownian \((B, S)\)-security market, including fractional one, is not possible since we have an incomplete market. Following the idea proposed by Föllmer and Sondermann (1986) and Föllmer and Schweizer (1993) we look for the strategy locally minimizing the risk. The strategy \(\pi^*\) is locally risk-minimizing, if for any \(H\)-admissible \(X_T = H\), where \(X_T\) is a capital at time \(T\) strategy \(\pi\) and any \(t\)

\[
R_t(\pi^*) \leq R_t(\pi),
\]

where the residual risk \(R_t(\pi)\) is defined as follows

\[
R_t(\pi) := E_x^P \left[ \frac{1}{2} \left( C_T(\pi) - C_t(\pi) \right)^2 / \mathcal{F}_t \right],
\]

and

\[
C_t(\pi) := X_t(\pi) - \int_0^t \beta_u \, dS_u,
\]

\(\beta_t\) is the number of stocks at time \(t\), \(E_x^P\) is the expectation with respect to a measure \(P\) conditionally \(x_0 = x\).

**Remark 2.** The residual risk process (see Remark 1, Section 2.4) is expressed in the following way

\[
R_t(\pi^*) = E_x^P \left. \Phi \left( \int_0^T \left[ Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r) Qu(r, S_r, x_r) \right] \, dr / \mathcal{F}_t \right) \right|_{x=x_t},
\]

where the function \(u\) satisfies the following boundary value problem (see Elliott & Chan (2004) and Bender (2003))

\[
\begin{align*}
& u_t(t, S, x) + ruS + H \sigma^2(x) \cdot S \sigma^{2H-1} \cdot u_S(t, S, x) - ru + Qu(t, S, x) = 0 \\
& u(T, S, x) = f(S).
\end{align*}
\]

In particular residual risk at the moment \(t = 0\) is equal to

\[
R_0(\pi^*) = E_x^P \left. \Phi \left( \int_0^T \left[ Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r) Qu(r, S_r, x_r) \right] \, ds \right) \right|_{x=x_t},
\]

where the operator \(Q\) is the infinitesimal operator of the process \(x_t\).
2.5 The Black-Scholes Formula for Markov-Modulated Fractional Brownian \((B, S)\)-Security Market with Jumps in the Hu & Øksendal Scheme

Let us consider fractional Markov-modulated \((B, S)\)-security market (7) on the intervals \([\tau_k, \tau_{k+1})\), but at the moment \(\tau_k\) we have the jump of \(S_t\):

\[
S_{\tau_k} - S_{\tau_k^-} = S_{\tau_k} u_k,
\]

where \(u_k, \ k \geq 1\), are independent i.d.r.v. with values in \((-1, +\infty)\) and distribution function \(H(dy)\). The moments \(\tau_k\) are the moments of jumps for the Poisson process \(N_t\) with intensity \(\lambda > 0\). We suppose that \(\tau_k, u_k\), are independent on \(x_t\) and \(B^H_t, \ k \geq 1\).

Let us denote by \(\mathcal{F}_t\) the \(\sigma\)-algebra generated by the r.v. \(B^H_t, N_t\), and \(u_j 1_{\{j \leq N_t\}}\) for \(j \geq 1\), where \(1_A = 1\), if \(\omega \in A\), and \(1_A = 0\), if \(\omega \notin A\).

It can be shown that \(B^H_t\) is a fractional Brownian motion w.r.t. \(\mathcal{F}_t\), and that \(N_t\) is a process adapted to this filtration and with \(N_t - N_s\) independent of the \(\sigma\)-algebra \(\mathcal{F}_t\) for all \(t > s\).

Taking into account (40) and the following representation

\[
S_T = S_0 e^{\int_0^T \sigma(x_u)dB^H_u + \int_0^T \mu(x_u)du - \frac{1}{2} \int_0^T \sigma(x_u)\sigma(x_s)\phi(s, u)dsdu},
\]

we obtain

\[
S_t = S_0 (1 + \sum_{j=1}^{N_t} (1 + u_j)) e^{\int_0^T \sigma(x_u)dB^H_u + \int_0^T \mu(x_u)du - \frac{1}{2} \int_0^T \sigma(x_u)\sigma(x_s)\phi(s, u)dsdu}
\]

with convention \(\prod_{j=1}^{0} = 1\).

Let

\[
\eta_{H,t} := e^{\int_0^T K^H_t(s, x_u)dB^H_s - \frac{1}{2} \int_0^T K^H_t(s, x_u)K^H_t(u, x_u)\phi(s, u)dsdu} \prod_{k=1}^{N_t} h(u_k),
\]

where \(h(y)\) is a function such that

\[
\begin{align*}
\int_{R} h(y) H(dy) &= 1, & \text{and} \\
\int_{R} y h(y) H(dy) &= 0,
\end{align*}
\]

where \(H(dy)\) is a distribution of \((u_k; \ k \geq 1)\) on \((-1, +\infty)\). We note, that \((\lambda, H(dy))\) is a \((P, \mathcal{F}_t)\)-local quadratic variation of the compound Poisson process \(\sum_{k=1}^{N_t} u_k\) independent of \(w_t\).

Let \(P_{H,\phi}^\ast\) be a measure such that

\[
\frac{dP_{H,\phi}^\ast}{dP} = \eta_{H,t}^\ast.
\]

\(P_{H,\phi}^\ast\) is a probability measure.

Let introduce the following measure:

\[
\frac{d\tilde{P}_{H,\phi}}{dP} = \eta_{H,t}^\ast \mathcal{E}_T^\phi.
\]

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We note that $P_{H,\phi}$ is a probability measure. Therefore, we have two distinct measures, namely, $P_{H,\phi}$ and $\hat{P}_{H,\phi}$, with respect to which we can calculate option pricing formula.

Hence, the $(B, S)$-security market with jumps is incomplete (see Section 2.3).

Using the representation
\[ S_T = S_0 \int_0^T \sigma(x_u) d B_u^H + \int_0^T \nu(x_u) d u - \frac{1}{2} \int_0^T \sigma(x_u) \sigma(x_u) \phi(s, u) d u, \]

Theorem 2 and Corollary 3 (see Elliott & Swishchuk (2005)) we immediately obtain the following result.

**Theorem 3.** Let $f_T(S) = (S - K)^+$, and $r(x) \equiv r$. Then the price of $C_0(x, S)$ of European call option has the form:

\[ C_0(x, S) = \sum_{k=0}^{+\infty} \frac{\exp \{-XT\} (kT)^k}{k!} \times \int_{-1}^{+\infty} \ldots \int_{-1}^{+\infty} \int_{-1}^{+\infty} \int_{-1}^{+\infty} C_B^S(\frac{z}{T})^{2-1}, T, S, \prod_{i=1}^{k}(1 + y_i)) F^T_t(dz) \]

\[ \times H^*(dy_1) \times \ldots \times H^*(dy_k), \]

where the function $C_B^S(\sigma, T, S)$ is a Black-Scholes value for the European call option in the Hu\&Oksendal scheme:

\[ C_B^S(\sigma, T, S) := S \Phi(d_+^H) - K e^{-rT} \Phi(d_-^H), \]

and

\[ d_\pm^H = [\log \frac{S_0}{K} + rT \pm T^2 H^2] / 2 T^2 H^2, \]

$H^*(dy) := h(y) H(dy)$ and $h(y)$ is defined in (32).

**Remark 3.** Perfect hedging in a Markov-modulated fractional Brownian $(B, S)$-security market with jumps is not possible since we have an incomplete market. We look for the strategy locally minimizing the risk.

The residual risk process (see Remark 1, Section 2.4) is expressed in the following way

\[ R_0(\pi) = E^{P_{H,\phi}} \left( \int_0^T [Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r)Qu(r, S_r, x_r)] dr / F_0 \right), \]

where the function $u(t, S, x)$ satisfies the following boundary value problem (see Remark 4)

\[ u_t(t, S, x) + rS u_S(t, S, x) + H \sigma^2(x) \cdot S^2 \partial^2 H^{-1} u_S(t, S, x) + \lambda \int_1^\infty [u(t, S(1 + v), x) - u(t, S, x)] H^*(dv) - ru + Qu(t, S, x) = 0, \]

\[ u(T, S, x) = f(S). \]

In particular, the residual risk at the moment $t = 0$ equals

\[ R_0(\pi) = E^{P_{H,\phi}} \left( \int_0^T [Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r)Qu(r, S_r, x_r)] ds \right), \]

where the operator $Q$ is the infinitesimal operator of the process $x_t$. 

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3 Pricing Options for General Markov-Modulated Fractional Brownian \((B, S)\)-Security Markets (Elliott & van der Hoek Scheme)

3.1 General Fractional \((B, S)\)-Security Markets (Elliott & van der Hoek Scheme (2000))

Let \((S^\ast(R), \mathcal{F})\), the space of tempered distributions, be the underlying probability space with \(\mathcal{F}\) the Borel sigma field. A probability measure \(P\) is given on \((S^\ast(R), \mathcal{F})\) by the Bochner-Minlos theorem.

We note that the general fractional \((B, S)\)-security market defined by Elliott & van der Hoek (2000) has two investment possibilities:

1) A bank account or a bond, where the price \(B_t\) at time \(t\) develops according to the equation

\[
 dB_t = r B_t \, dt, \quad B_0 > 0, \quad r > 0; \tag{35}
\]

2) A stock, where the price \(S_t\) at time \(t\) satisfies the equation

\[
 dS_t = \mu S_t \, dt + \sigma S_t B_M(t), \quad S_0 > 0, \quad \mu \in \mathbb{R}, \quad \sigma > 0, \quad t \in [0, T], \tag{36}
\]

where the process

\[
 B_M(t) := \sigma_1 B_{H_1}(t) + \sigma_2 B_{H_2}(t) + \ldots + \sigma_m B_{H_m}(t), \quad 0 < H_k < 1, \quad \sigma_k > 0, \quad k = 1, 2, \ldots, m, \tag{37}
\]

\(B_H\) are fractional Brownians motion with Hurst index \(H_k \in (0, 1)\), the operator

\[
 M := \sigma_1 M_{H_1} + \ldots + \sigma_m M_{H_m}, \tag{38}
\]

where \(M_H\) is a fundamental operator: \(\forall f \in S(\mathbb{R})\):

\[
 M_H f(x) := \begin{cases} 
 (2\Gamma(H - \frac{1}{2}) \cos(\frac{\pi}{2} (H - \frac{1}{2})))^{-1} \int_{\mathbb{R}} \frac{f(x) - f(x + dt)}{|t|^{\frac{3}{2} - H}} \, dt, & \text{if } H \in (0, 1/2), \\
 (2\Gamma(H - \frac{1}{2}) \cos(\frac{\pi}{2} (H - \frac{1}{2})))^{-1} \int_{\mathbb{R}} \frac{f(x + dt) - f(x)}{|t|^{\frac{3}{2} - H}} \, dt, & \text{if } H \in (1/2, 1), \\
 f(x), & \text{if } H = \frac{1}{2}.
\end{cases} \tag{39}
\]

From [4] we know that

\[
 S_t = S_0 e^{\mu t + \sigma B_M(t) - \frac{\sigma^2 M(t)}{2}}, \tag{40}
\]

where

\[
 \sigma^2_M(t) := \sum_{i,j=1}^m 2(\sin(\frac{\pi}{2} (H_i + H_j))) T(H_i + H_j + 1))^{-1} |t|^{H_i + H_j}. \tag{41}
\]

As was stated in Elliott and van der Hoek (2000), this general fractional \((B, S)\)-security market is complete.
3.2 General Markov-Modulated Fractional Brownian 
\((B, S)\)-Security Markets in the Elliott & van der Hoek Scheme

Let us consider the general fractional \((B, S)\)-security market:

\[
\begin{align*}
\frac{dB_t}{dt} &= r(x_t)B_t dt, \quad B_0 > 0, \\
\frac{dS_t}{dt} &= S_t(\mu(x_t)dt + \sigma(x_t)dB_M(t)), \quad S_0 > 0,
\end{align*}
\]

where \(x_t\) is a homogeneous Markov process independent of \(B_M(t)\) with infinitesimal operator \(Q\), \(\mu(x)\) and \(\sigma(x)\) are continuous and bounded functions on the phase space of states \(X\), and the process \(x_t\) is independent of \(B_M(t)\). We note that \(S_t\) in (76) may be written in another form:

\[
\frac{dS_t}{dt} = \mu(x_t)S_t + \sigma(x_t)S_t \hat{\diamond} W_M(t),
\]

where \(W_M(t)\) is the \(M\)-fractional white noise, \(W_M(t) \in S^*\), namely, \(W_M(t)\) is the derivative in \(S^*\) of \(B_M(t)\) (see Aase, K., Øksendal, B. and Øboe, J. (1998) and Hu, Y. and Øksendal, B. (1999)) and \(\hat{\diamond}\) is a Wick product.

Using Wick calculus (see Hu & Øksendal (1999), Elliott & van der Hoek (2000)) we see that the solution of the last equation takes the form:

\[
S_t = S_0\exp\left(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)W_M(s)ds\right)
= S_0\exp\left(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)dB_M(s)\right)
= S_0\exp\left(\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)dB_M(s) - \frac{1}{2} \int_0^t \int_0^t \sigma(x_s)\sigma(x_u)M_H(s)M_H(u)dsdu\right),
\]

where \(M_H\) is defined in (73).

3.3 Incompleteness of General Markov-Modulated Fractional Brownian \((B, S)\)-Security Markets

Let

\[
\hat{B}_M(t) := \int_0^t \frac{\mu(x_s) - r(x_s)}{\sigma(x_s)} ds + B_M(t).
\]

By the Girsanov theorem (given by (4.2) in Elliott and van der Hoek (2000)), \(\hat{B}_M(t)\) is a fractional Brownian motion with respect to the measure \(\hat{P}\) defined in \(\mathcal{F}\) by

\[
\frac{d\hat{P}_M^\phi}{dP} = \exp[\phi, \xi > -\frac{1}{2}||\phi||^2] := \eta^M_\phi,
\]

where

\[
\phi(t) = \int_0^t \frac{\mu(x_s) - r(x_s)}{\sigma(x_s)} ds M^{-1}(I(0, T)).
\]

In particular, if \(M = M_H\), then

\[
\phi(t) = \int_0^t \frac{\mu(x_s) - r(x_s)}{\sigma(x_s)} ds \left[ \frac{(T-t)}{T-t}^{3/2-\bar{H}} + \frac{t}{t^{3/2-\bar{H}}} \right].
\]
Consequently, under $\hat{P}_\phi^M$, $S_t$ has dynamics:

$$dS_t = r(x_t)S_t dt + \sigma(x_t)S_t dB_M(t).$$

Let $\pi_t = (\alpha_t, \beta_t)$ be a portfolio, namely, a pair of $\mathcal{F}_t^M (:= \sigma\{B_M(s); s \in [0, t]\})$ adapted processes which gives the number of units of $B_t$ and $S_t$, respectively, held at time $t$.

The corresponding wealth process is

$$X_t^\pi = \alpha_t B_t + \beta_t S_t.$$

The market is self-financing, if

$$dX_t^\pi = \alpha_t dB_t + \beta_t dS_t.$$

We will assume that our market is self-financing.

Note that

$$X_t^\pi = \frac{X_0^\pi - \beta_t S_t}{B_t},$$

so

$$dX_t^\pi = r(x_t)X_t^\pi dt + \sigma(x_t)\beta_t S_t d\hat{B}_M(t).$$

The usual argument now shows that our market is arbitrage free:

$$\exp\{-\int_0^t r(x_s) ds\} X_t^\pi = X_0^\pi + \int_0^t \exp\{-\int_0^s r(x_u) du\} \sigma(x_s)\beta_s S_s d\hat{B}_M(s), \quad t \in [0, T],$$

and

$$E_{\hat{P}_\phi^M}\left[\exp\{-\int_0^T r(x_s) ds\} X_T^\pi\right] = X_0^\pi = X.$$

Therefore, there cannot be a portfolio $\pi_t = (\alpha_t, \beta_t)$ for which $X_0^\pi \leq 0$ and $X_T^\pi > 0$ with $P(X_T^\pi > 0) > 0$.

Suppose that $F(\omega)$ is an $\mathcal{F}_T^M$-measurable r.v. in $L^2_M$. We can construct a portfolio $\pi_t = (\alpha_t, \beta_t)$ and an investment such that $F(\omega) = X_T^\pi$ a.s. This is the same as requiring

$$\exp\{-\int_0^T r(x_s) ds\} F(\omega) = X_0^\pi + \int_0^T \exp\{-\int_0^s r(x_u) du\} \sigma(x_s)\beta_s S_s d\hat{B}_M(s), \quad t \in [0, T].$$

However, applying the fractional representation result [4] with $B_M$ replaced by $\hat{B}_M$, we have

$$\exp\{-\int_0^T r(x_s) ds\} F(\xi) = E_{\hat{P}_\phi^M}\left[\exp\{-\int_0^T r(x_s) ds\} F\right] + \int_0^T E_{\hat{P}_\phi^M}\left[\exp\{-\int_0^s r(x_u) du\} \hat{D}_t^M f / \mathcal{F}_t^M\right] d\hat{B}_M(s), \quad t \in [0, T].$$

Here, $\hat{D}_t^M$ is the stochastic $M$-gradient under measure $\hat{P}_\phi^M$.

Let us introduce the following measure

$$\frac{\hat{P}_\phi^M}{dP} = \eta^M \mathcal{E}_T^\pi,$$  \hspace{1cm} (46)
where $\gamma^M$ and $\mathcal{E}_T^\pi$ are defined in (45) and (13), respectively.

Using the same reasoning as just above, we obtain that

$$E_{\hat{P}_\phi}[\exp\{-\int_0^t r(x_s)ds\}X_t^\pi] = X_0^\pi = X.$$  

Thus, we can conclude that we have two measures $\hat{P}_\phi^M$ and $\hat{P}_\phi^M$ in (45) and (46), respectively. Since we have two measures, the market is incomplete.

We can use the martingale measure $\hat{P}_\phi^M$ to calculate the price of the claim $F(\omega)$.

Summarizing all the above results we obtain the following result.

**Theorem 8.** The general fractional $(B, S)$-security market has no arbitrage. It is incomplete and

$$X_0^\pi = X = E_{\hat{P}_\phi}[\exp\{-\int_0^T r(x_s)ds\}F]$$  

(47)

is the price of the claim $F$.

The following functions

$$\alpha_t = \frac{X_T - \beta_t S_t}{\hat{P}_\phi},$$

$$\beta_t = S_t^{-1} \exp\{-\int_0^{T-t} r(x_s)ds\} \sigma^{-1}(x_t) E_{\hat{P}_\phi}[\exp\{-\int_0^T r(x_s)ds\} \hat{D}_t^M F/\mathcal{F}_t^M]$$

give the portfolio.

### 3.4 A Black-Scholes Formula for a General Markov-Modulated Fractional Brownian $(B, S)$-Security Market

Suppose $F(\omega)$ is a European call

$$F(\omega) = (S_T(\omega) - K)^+,$$

with strike price $K$, and $r(x) = r, \ \forall x \in X$. From Theorem 4, formula (45) and formula (47) we obtain the price $C^M_0(x, S)$ for the European call option at time 0.

**Theorem 5.**

$$C^M_0(x, S) = \int C^M_{BS}(\left(\frac{z}{\sigma^2_M(T)}\right)^{1/2}, T) F^*_T(dz),$$

where

$$C^M_{BS}(\sigma, T, S) := S \Phi(d^M_+) - Ke^{-rT} \Phi(d^M_-),$$

$$d^M_+ := [\log \frac{S_0}{K} + rT + \frac{\sigma^2 M(T)}{2}] / \sigma_M(T),$$

$$d^M_- := [\log \frac{S_0}{K} + rT - \frac{\sigma^2 M(T)}{2}] / \sigma_M(T),$$

$$F^*_T(dz) := P\{\int_0^T \sigma(x)\sigma(x_u)M_T(s)M_T(u)dsdu \in (z, z + dz)\},$$

$$\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{x^2}{2}} dx.$$  

**Remark 4.** Perfect hedging in a general Markov-modulated Brownian $(B, S)$-security market is not possible since we have an incomplete market. We look for the strategy locally minimizing the risk.

The residual risk process (see Remark 1, Section 2.4) is expressed in the following way

$$R_t(\pi^*) = E_{\hat{P}_\phi}^M \left( \int_t^T [Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r)Qu(r, S_r, x_r)] \, dr / F_t \right),$$

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where the function \( u \) satisfies the following boundary value problem:

\[
\begin{align*}
  u_t(t, S, x) + r u_S(t, S, x) + H \sigma^2(x) \cdot S^{2 \sigma M(t)} u_{SS}(t, S, x) - r u + Qu(t, S, x) & = 0 \\
  u(T, S, x) & = f(S).
\end{align*}
\]

In particular residual risk at the moment \( t = 0 \) equals to

\[
R_0(\pi^*) = E_x^\pi \left( \int_0^T [Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r)Qu(r, S_r, x_r)] \, ds \right),
\]

where \( Q \) is the infinitesimal operator of the process \( x_t \).

### 3.5 A Black-Scholes Formula for a General Fractional Markov Modulated Brownian \((B, S)\)-Security Market with Jumps in Elliott & van der Hoek Scheme

Let us consider fractional Markov-modulated \((B, S)\)-security market (76) on the intervals \([\tau_k, \tau_{k+1}]\), but at the moment \( \tau_k \) we have the jump of \( S_t \):

\[
S_{\tau_k} - S_{\tau_-} = S_{\tau_k} u_k,
\]

where \( u_k, \ k \geq 1, \) are independent i.d.r.v. with values in \((-1, +\infty)\) and distribution function \( H(dy) \). The moments \( \tau_k \) are the moments of jumps for the Poisson process \( N_t \) with intensity \( \lambda > 0 \). We suppose that \( \tau_k, u_k, \) are independent on \( x_t \) and \( B_M(t), \ \ k \geq 1 \).

Let us denote by \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by the r.v. \( B_M(t), \ N_t, \) and \( u_j 1_{(j \leq N_t)} \) for \( j \geq 1, \) where \( 1_A = 1, \) if \( \omega \in A, \) and \( 1_A = 0, \) if \( \omega \notin A. \)

It can be shown that \( B_M(t) \) is a fractional Brownian motion w.r.t. \( \mathcal{F}_t, \) and that \( N_t \) is a process adapted to this filtration and \( N_t - N_s \) is independent of the \( \sigma \)-algebra \( \mathcal{F}_t \) for all \( t > s. \)

Taking into account (77) and the following representation

\[
S_t = S_0 \exp(\int_0^t \mu(x_s) \, ds + \int_0^t \sigma(x_s) dB_M(s) - \frac{1}{2} \int_0^t \sigma(x_s)^2 dB_M(s)) - \frac{1}{2} \int_0^t \sigma(x_s) dB_M(s)\sigma(x_u) M_H(s) M_H(u) ds du,
\]

we obtain

\[
S_t = S_0 \left( \prod_{j=1}^{N_t} (1 + u_j) \right) \exp^\frac{1}{2} \int_0^t \mu(x_s) ds + \int_0^t \sigma(x_s) dB_M(s) - \frac{1}{2} \int_0^t \sigma(x_s)^2 dB_M(s)\sigma(x_u) M_H(s) M_H(u) ds du,
\]

with convention \( \prod_{j=1}^{0} = 1. \)

Let

\[
\eta_{M,t} := e^{-<\phi, \xi> + \frac{1}{2} \|\phi\|^2} \prod_{k=1}^{N_t} i(h(u_k)),
\]

where \( \phi(t) \) is defined in (45), and \( h(y) \) is a function such that

\[
\left\{
\begin{array}{ll}
\int_R h(y) H(dy) &= 1, & \text{and} \\
\int_R y h(y) H(dy) &= 0,
\end{array}
\right.
\]

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where $H(dy)$ is a distribution of $(u_k; \ k \geq 1)$ on $(-1, +\infty)$. We note, that $(\lambda, H(dy))$ is a $(P, \mathcal{F}_t)$-local quadratic variation of the compound Poisson process $\sum_{k=1}^{N_t} u_k$ independent of $u_t$.

Let $P_{M,\phi}^*$ be a measure such that

$$\frac{dP_{M,\phi}^*}{dP} = \eta_{M,T}^*.$$

$P_{H,\phi}^*$ is a probability measure.

Let us also introduce the following measure

$$\frac{dP_{M,\phi}}{dP} = \eta_{M,T}^* E^\sigma_T.$$

We note, that $P_{M,\phi}$ is a probability measure.

Therefore, we have two distinct measures, namely, $P_{M,\phi}$ and $P_{M,\phi}^*$ (Section 3.3) with respect to which we can calculate option pricing formula.

Hence, the $(B, S)$-security market with jumps is incomplete (see Section 3.3).

Using the representation

$$S_t = S_0(\prod_{j=1}^{N_t}(1 + u_j))e^{\int_0^t \mu(x_s)ds + \int_0^t \sigma(x_s)dM_t(s) - \frac{1}{2} \int_0^t \sigma(x_s)^2 M_t(s) M_t(u) ds du},$$

Theorem 2 and Corollary 3 (see Elliott & Swishchuk (2005)) we obtain immediately the following result.

**Theorem 6.** Let $f_T(S) = (S - K)^+$, and $r(x) \equiv r$. Then the price of European call option $C_0^M(x, S)$ has the form:

$$C_0^M(x, S) = \sum_{k=0}^{\infty} \frac{\exp\{-\lambda T\}(\lambda T)^k}{k!} \times \int_{-1}^{+\infty} \ldots \int_{-1}^{+\infty} \int C_T^{BS}((\tilde{z}_T)^{2^{-1}}, T, S, \prod_{i=1}^{k}(1 + y_i)) E \Phi^\sigma_T(dz)$$

$$\times H^*(dy_1) \times \ldots \times H^*(dy_k),$$

where the function $C_T^{BS}(\tilde{\sigma}, T, S)$ is a Black-Scholes value for European call option in the Elliott & van der Hoek scheme:

$$C_0^M(\tilde{\sigma}, T, S) := S \Phi(d_+^M) - K e^{-rT} \Phi(d_-^M),$$

and

$$d_{+}^M = \left\{ \log \frac{S_0}{K} + rT \pm \frac{\sigma^2}{2} \sigma_{M}(T) \right\}/\tilde{\sigma} \sigma_{M}(T),$$

$$F^\sigma_T(dz) := P\{\int_0^T \int_0^T \sigma(x_s) \sigma(x_u) M_T(s) M_T(u) ds du \in (z, z + dz)\},$$

$$\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} dx.$$

**Remark 5.** Perfect hedging in a general Markov-modulated fractional Brownian $(B, S)$-security market with jumps is not possible since we have an incomplete market. We look for the strategy locally minimizing the risk.
The residual risk process (see Remark 1, Section 2.4) is expressed in the following way
\[ R_t(\pi^*) = E_x^{\pi^*}(\int_t^T [Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r)Qu(r, S_r, x_r)] \, dr / F_t), \]
where the function \( u \) satisfies the following boundary value problem
\[
\begin{align*}
  u_t(t, S, x) &+ r Su_s(t, S, x) + H\sigma^2(x) \cdot S^2 \frac{\mu(t)}{r} u_{ss}(t, S, x) \\
  &+ \lambda \int_1^\infty [u(t, S(1 + v), x) - u(t, S, x)]H^*(dv) - ru + Qu(t, S, x) = 0
\end{align*}
\]
\[ u(T, S, x) = f(S), \]

\( H^*(dy) := h(y)H(dy) \), where \( h(y) \) is defined in (32).

In particular residual risk at the moment \( t = 0 \) is equal to
\[ R_0(\pi^*) = E_x^{\pi^*}(\int_0^T [Qu^2(r, S_r, x_r) - 2u(r, S_r, x_r)Qu(r, S_r, x_r)] \, ds), \]

where the operator \( Q \) is the infinitesimal operator of the process \( x_t \).

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