Functional Monetary Aggregates, Monetary Policy, and Business Cycles*

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In this paper, we take the economic approach to monetary aggregation. In the context of highly disaggregated demand systems, encompassing the full range of monetary assets in the United States, we estimate the effectively globally regular minflex Laurent (ML) and normalized quadratic (NQ) flexible functional forms. We produce the ML and NQ functional monetary aggregates, consistent with neoclassical microeconomic theory, also addressing the issue of optimal monetary aggregation. We highlight the influence of measurement on statistical inference by investigating whether the ML and NQ functional monetary aggregates are of importance in resolving paradoxes associated with the measurement of money. In doing so, we also provide a comparison between the functional monetary aggregates and the Fed's (broad) Sum M2 aggregate and the Center for Financial Stability (broad) Divisia M3 and Divisia M4 aggregates. Our detailed statistical analysis favors the functional monetary aggregates.

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1 Introduction

Currently, the common practice among central banks is to use 'simple-sum' aggregation to construct money measures from a list of possible components that are considered to be the likely sources of monetary services, as follows

$$M = \sum_{j=1}^{n} x_j$$

where M is the monetary aggregate and x_j is one of the n monetary components of the monetary aggregate. However, Friedman and Schwartz (1970, p. 151–152) dismissed simple-sum monetary aggregates, arguing that "this (summation) procedure is a very special case of the more general approach. In brief, the general approach consists of regarding each asset as a joint product having different degrees of *moneyness*, and defining the quantity of money as the weighted sum of the aggregate value of all assets, the weights for individual assets varying from zero to unity with a weight of unity assigned to that asset or assets regarded as having the largest quantity of 'moneyness' per dollar of aggregate value. The procedure we have followed implies that all weights are either zero or unity."

Over the years, there have been many attempts at properly weighting the monetary components within a simple-sum aggregate, but without theory, any weighting scheme is questionable. Barnett (1980) argued instead for applying aggregation theory and statistical index number theory to monetary aggregation. He also argued [see Barnett (1980, p. 12)] that "[w]hile aggregation theory results in exact aggregator functions depending upon unknown (but estimable) parameters, statistical index number theory results in parameter-free approximations to aggregator functions. Index number theory provides the basis for the index numbers published by nearly every governmental agency in the world (other than the central banks)." In this regard, statistical index number theory provides a class of quantity and price indexes that can be computed from price and quantity data alone, thus eliminating the need to estimate an underlying structure. Statistical indexes are mainly characterized by their statistical properties. These properties were examined in great detail by Fisher (1922) and serve as tests in assessing the quality of statistical indexes. While Fisher (1922) found the simple-sum index to be the worst known index number formula, the index that he found to be the best has now become known as the Fisher ideal index, which is the geometric average of the Laspeyres and Paasche indexes. Another index found to possess a very large number of such properties is the (Törnqvist) discrete time approximation to the continuous Divisia (1925) index; Divisia (1925) proposed the continuous time index for aggregating over goods.

Barnett (1978, 1980) proved how the Divisia approach to aggregation could be extended to include monetary assets and constructed monetary quantity indexes, now known as Divisia monetary aggregates. The Divisia index (in discrete time) is defined by

$$\log M_t^D - \log M_{t-1}^D = \sum_{j=1}^{n} w_{jt}^* (\log x_{jt} - \log x_{j,t-1})$$

according to which the growth rate of the aggregate is the weighted average of the growth rates of the component quantities, with the weights being defined as the expenditure shares averaged over the two periods of the change, $w_{jt}^* = (1/2)(w_{jt} + w_{j,t-1})$ for $j = 1, ..., n$, where $w_{jt} = \pi_{jt}x_{jt}/\sum_{k=1}^{n} \pi_{kt}x_{kt}$ is the expenditure share of asset j during period t, and π_{jt} is the user cost of asset j, derived in Barnett (1978)

$$\pi_{jt} = \frac{(R_t - r_{jt})}{(1 + R_t)}$$

which is just the opportunity cost of holding a dollar's worth of the jth asset. In the equation above, r_{jt} is the market yield on the jth asset and R_t is the yield available on a 'benchmark' asset that is held only to carry wealth between multiperiods — see Barnett *et al.* (1992), Barnett and Serletis (2000), or Barnett (2012) for more details regarding the Divisia approach to monetary aggregation. Barnett has also extended the field of index number theory to include risk in Barnett (1995), Barnett *et al.* (1997), and Barnett and Wu (2005). He extended index number theory to multilateral international financial aggregation in Barnett (2007), for multicountry economic unions. More recently, Barnett *et al.* (2016) further extended the Divisia monetary aggregates to the credit card-augmented Divisia monetary aggregates, which jointly account for the liquidity services provided by monetary assets and credit cards.

Over the years a large number of articles have shown that the use of the Divisia monetary aggregates can solve the "Barnett critique"— the measurement problems associated with the failure to find significant relations between money and key macroeconomic variables. See, for example, Barnett and Chauvet (2011), Hendrickson (2014), Serletis and Gogas (2014), Belongia and Ireland (2014, 2015, 2016, 2018), Ellington (2018), Dai and Serletis (2020), and Dery and Serletis (2020), among others. In fact, Belongia and Ireland (2015, p. 268) "call into question the conventional view that the stance of monetary policy can be described with exclusive reference to its effects on interest rates and without consideration of simultaneous movements in the monetary aggregates." They argue that properly measured monetary aggregates, such as the new Center for Financial Stability (CFS) Divisia monetary aggregates, can and should play an important role (either as intermediate targets or indicator variables) for the conduct of monetary policy, in addition to that of the short-term nominal interest rate.

The fields of aggregation theory and statistical index number theory developed independently. However, Diewert (1976) provided the link between economic and aggregation theory and statistical index number theory by attaching economic properties to statistical indexes. These properties are defined in terms of the ability of a statistical index to approximate a particular functional form for the unknown underlying aggregator function. In fact, for a number of well known statistical indexes Diewert (1976) shows that they are equivalent to the use of a particular functional form. Such statistical indexes are called 'exact.' Exactness, however is not sufficient for acceptability of a particular statistical index when the functional form for the aggregator function is not known. In this case it seems desirable to choose a statistical index which is exact for a flexible functional form. Diewert termed such statistical indexes 'superlative.' Diewert also showed that the Divisia index is exact for the linearly homogeneous translog flexible functional form, and is, therefore, superlative.

The translog flexible functional form, introduced by Christensen *et al.* (1975), is a locally flexible functional form (a second-order local approximation to an arbitrary function), and as Caves and Christensen (1980) and Barnett and Lee (1985) have shown the regularity regions of locally flexible functional forms can be relatively small. However, over the years, an increasingly sophisticated literature has been under development on flexible functional forms. There is now a large number of other locally flexible functional forms, including the generalized Leontief (GL), introduced by Diewert (1973), and the Almost Ideal Demand System (AIDS), introduced by Deaton and Muellbauer (1980). There are also the effectively globally regular minflex Laurent (ML) models, which are based on the Laurent series expansion and were introduced by Barnett (1983) and Barnett and Lee (1985), the quadratic AIDS (QUAIDS) model of Banks *et al.* (1997), the general exponential form (GEF) of Cooper and McLaren (1996), and the normalized quadratic (NQ) models, introduced by Diewert and Wales (1988). Also, the globally flexible Fourier and Asymptotically Ideal Model (AIM) models, introduced by Gallant (1981) and Barnett and Jonas (1983), respectively, could be used.

This raises interesting methodological questions. If the Divisia index is exact to the linearly homogeneous translog flexible functional form, which has a relatively small regular region, can we use a statistical index from the statistical index number literature that is exact to a flexible functional form that has a larger regularity region? We are not aware of statistical indexes that can be shown to be exact to the effectively globally regular flexible functional forms or the globally flexible functional forms. For this reason, in this paper we build on a large body of literature, which Barnett (1997) calls the 'high road' literature, and take a microeconomic- and aggregation-theoretic approach to the demand for monetary assets and monetary aggregation. The approach that we take allows the estimation in a systems context assuming a flexible functional form for the aggregator function, based on the dual approach to demand system generation developed by Diewert (1974). We estimate two popular, effectively globally regular flexible functional forms, the minflex Laurent and normalized quadratic, to produce the ML and NQ functional monetary aggregates. We do so, in the context of highly disaggregated demand systems, encompassing the full range of monetary assets, unlike earlier work in this area that has generally been carried out in the context of small, highly aggregated demand systems.

In doing so, we pay explicit attention to theoretical regularity, since the usefulness of flexible functional forms depends on whether they satisfy the theoretical regularity conditions of positivity, monotonicity, and curvature, and in the older monetary demand systems literature there has been a tendency to ignore regularity. In fact, as Barnett (2002, p. 199) put it in his *Journal of Econometrics* Fellow's opinion article, without satisfaction of all three theoretical regularity conditions " ... the second-order conditions for optimizing behavior fail, and duality theory fails. The resulting first-order conditions, demand functions, and supply functions become invalid." Motivated by these considerations, we treat the curvature property as a maintained hypothesis in order to produce functional monetary aggregates consistent with neoclassical microeconomic theory and aggregation theory. We argue that unless regularity is attained by luck, flexible functional forms should always be estimated subject to regularity, as suggested by Barnett (2002).

We also address the issue of optimal monetary aggregation by (assuming that money has positive value in equilibrium, implying the existence of a monetary services aggregator function and) estimating large demand systems, encompassing the full range of monetary assets, to produce broad functional monetary aggregates. In this regard, Jadidzadeh and Serletis (2019) also address the issue of optimal monetary aggregation in the context of a large demand system. They provide evidence, based on disaggregated monetary demand responses, that the simple-sum monetary aggregates used by central banks around the world are inconsistent with neoclassical microeconomic theory. Their statistical tests also reject the necessary and sufficient conditions for all the money measures published by the Federal Reserve as well as a large set of null hypotheses that would be consistent with the existence of subaggregates of various subsets of liquid assets. Their tests support and reinforce Barnett's (2016) assertion that we should use, as a measure of money, the broadest Divisia M4 monetary aggregate prepared by the Center for Financial Stability.

Finally, we highlight the influence of measurement on statistical inference by investigating whether the ML and NQ (broad) functional monetary aggregates are of importance in resolving paradoxes associated with the measurement of money, in solving the Barnett critique, and in understanding the effects of potential monetary policy actions. We do so in the context of two dynamic general equilibrium monetary business models, the Ireland (2004) and Andrés *et al.* (2006) models, both of which find a minimal role of money in business cycle analysis. We also test for Granger causality from the ML and NQ functional monetary aggregates to industrial production, using the conventional VAR approach as well as the Psaradakis *et al.* (2005) Markov-switching model with time-varying parameters. We also provide a comprehensive comparison between the functional aggregates, ML and NQ, and the Fed's Sum M2 aggregate and the CFS broad Divisia M3 and Divisia M4 aggregates.

The rest of the paper is organized as follows. Sections 2 and 3 briefly sketch related neoclassical demand theory and aggregation theory. Section 4 presents the ML and NQ demand systems and discusses related econometric issues, paying explicit attention to the singularity problem and the imposition of global concavity. Section 5 discusses the data and presents the broad ML and NQ functional monetary aggregates. Section 6 presents the first group of empirical results and Section 7 presents the Granger causality test results. The final section concludes regarding the implications of our research for monetary theory, the conduct of monetary policy, and business cycle analysis.

2 Neoclassical Demand Theory

Let's consider an economy with identical households whose direct utility function is weakly separable (a direct tree) as follows

$$U = f\left(\boldsymbol{c},\, \ell,\, u\left(\boldsymbol{x}\right)\right) \tag{1}$$

where $\boldsymbol{c}$ is a κ-vector of consumption goods, l is leisure time, and $\boldsymbol{x}$ is a n-vector of the services of monetary assets (assumed to be proportional to the stocks). The utility-tree structure (1) is treated as a maintained hypothesis in this paper, as is the case with a large number of studies in the literature. It implies that the demand for monetary services is independent of relative prices outside the monetary group

$$\partial\left[\left(\partial u/\partial x_i\right)/\left(\partial u/\partial x_j\right)\right]/\partial c_k = 0, \text{ for } k = 1, ..., \kappa, \quad \text{and} \quad \partial\left[\left(\partial u/\partial x_i\right)/\left(\partial u/\partial x_j\right)\right]/\partial \ell = 0$$

where the expression in brackets is the marginal rate of substitution between monetary assets i and j.

Under the assumptions made, we can focus on the details of the demand for the services of monetary assets, ignoring the services of consumption goods, $\boldsymbol{c}$, and leisure, ℓ, in terms of the following consumer problem

$$\max_{\boldsymbol{x}} \quad u(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{p}'\boldsymbol{x} = y \tag{2}$$

where y is the expenditure on the services of monetary assets and $\boldsymbol{p} = (p_1, ..., p_n)'$ is the vector of monetary asset nominal user costs, with the jth element given as in Barnett (1978) by

$$p_j = P\frac{R - r_j}{1 + R}, \quad j = 1, ..., n$$

where P is the aggregate price level, R is the yield on an alternative asset (called benchmark asset), and r_j is the yield on the jth asset.

The solution of the first-order conditions for utility maximization is the demand system

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p}, y).$$

and the indirect utility function $h(\boldsymbol{p}, y) = u[\boldsymbol{x}(\boldsymbol{p}, y)]$.

2.1 The Indirect Utility Function

As already noted, the maximum level of utility at given prices and income, $h(\boldsymbol{p}, y) = u[\boldsymbol{x}(\boldsymbol{p}, y)]$, is the indirect utility function. The direct utility function and the indirect utility function are equivalent representations of the underlying preference preordering. Using $h(\boldsymbol{p}, y)$, we can derive the demand system by straightforward differentiation, without having to solve a system of simultaneous equations, as would be the case with the direct utility function first order conditions. In particular, Roy's identity

$$\boldsymbol{x}(\boldsymbol{p}, y) = -\frac{\partial h(\boldsymbol{p}, y)/\partial \boldsymbol{p}}{\partial h(\boldsymbol{p}, y)/\partial y} \tag{3}$$

allows us to derive the demand system, provided there is an interior solution and that $\boldsymbol{p} > \boldsymbol{0}$ and $y > 0$. Alternatively, the logarithmic form of Roy's identity

$$\boldsymbol{s}(\boldsymbol{p}, y) = -\left(\partial \log h(\boldsymbol{p}, y)/\partial \log \boldsymbol{p}\right) / \left(\partial \log h(\boldsymbol{p}, y)/\partial \log y\right)$$

or Diewert's (1974, p. 126) modified version of Roy's identity

$$s_j(\boldsymbol{v}) = \frac{v_j \nabla h(\boldsymbol{v})}{\boldsymbol{v}' \nabla h(\boldsymbol{v})} \tag{4}$$

can be used to derive the demand system in budget share form $\boldsymbol{s} = (s_1, ..., s_n)'$, where $s_j = p_j x_j(\boldsymbol{p}, y)/y$ is the expenditure share of asset j, $\boldsymbol{v} = (v_1, ..., v_n)'$ is a vector of expenditure normalized prices, with the jth element being $v_j = p_j/y$, and $\nabla h(\boldsymbol{v}) = \partial h(\boldsymbol{v})/\partial \boldsymbol{v}$.

The indirect utility function, $h(\boldsymbol{p}, y)$, is continuous in $(\boldsymbol{p}, y)$ and has the following properties: (i) positivity; (ii) homogeneity of degree zero in $(\boldsymbol{p}, y)$; (iii) decreasing in $\boldsymbol{p}$ and increasing in y; (iv) strictly quasi-convex in $\boldsymbol{p}$; and (v) satisfies Roy's identity. Together, properties (i)-(iv) are called the 'regularity conditions.' In the terminology of Caves and Christensen (1980), an indirect utility function is 'regular' at a given $(\boldsymbol{p}, y)$, if it satisfies the above properties at that $(\boldsymbol{p}, y)$. Similarly, the 'regular region' is the set of prices and income at which an indirect utility function satisfies the regularity conditions.

2.2 Functional Monetary Aggregates

Using a specific and differentiable form for the indirect utility function, $h(\boldsymbol{p}, y)$, and applying Roy's identity, we can derive the demand system. Using the derived demand system and specific monetary data, we then could estimate the parameters and replace the unknown parameters of $h(\boldsymbol{p}, y)$ by their estimates. The resulting estimated function is called an *economic* (or *functional*) monetary index, and its calculated value at any point in time is an economic monetary-quantity index number.

The problem is that the use of a specific functional form for the indirect utility function, $h(\boldsymbol{p}, y)$, necessarily implies a set of implicit assumptions about the underlying preference structure of the economic agent. For example, the use of a weighted linear aggregator function, implies perfect substitutability among the assets. The use of a Cobb-Douglas functional form imposes an elasticity of substitution equal to unity between every pair of assets. Similarly, a constant elasticity of substitution functional form, although it relaxes the unitary elasticity of substitution restriction imposed by the Cobb-Douglas, it imposes the restriction that the elasticity of substitution between any pair of assets is always constant.

For many years, the literature concentrated on the use of globally regular functional forms, such as the Cobb-Douglas and the constant elasticity of substitution functional forms. These forms globally satisfy the theoretical regularity conditions for rational neoclassical economic behavior. However, most members of this class of functions should be rejected, partly because of the restrictive nature of their implicit assumptions, and partly because of the existence of attractive alternatives. Among the alternatives are the flexible functional forms that have been developed, since the publication of Diewert's (1971) seminal paper on duality, to approximate unknown aggregator functions such as our indirect utility function, $h(\boldsymbol{p}, y)$.

Three popular flexible functional forms are the generalized Leontief, introduced by Diewert (1973), the translog, introduced by Christensen *et al.* (1975), and the almost ideal demand system of Deaton and Muellbauer (1980). As noted by Barnett and Serletis (2008), these locally flexible models provide the ability to attain arbitrary elasticities of substitution, although at only one point. However, as argued by Caves and Christensen (1980) and Barnett and Lee (1985), most popular locally flexible functional forms have very small regions of theoretical regularity. These models thereby violate the conditions for the duality theory from which the models were derived, except at points within the regular region.

A result was the development of, what Cooper and McLaren (1996) classify as, 'effectively globally regular' flexible functional forms, that is, locally flexible functional forms that have large (but not global) regular regions. These functions typically have regular regions that include almost all data points in the sample. In addition, the regularity regions increase as real expenditure levels grow, as is often the case with time series data. Examples of these functions include Barnett's minflex Laurent (ML) models, based on the Laurent series expansion, and the quadratic AIDS (QUAIDS) model of Banks *et al.* (1996). Moreover, Diewert and Wales (1988) proposed two locally flexible models for which the theoretical curvature conditions can be imposed globally — the first system is derived from a normalized quadratic (NQ) reciprocal indirect utility function and the second is derived from a NQ expenditure function.

Finally, a path-breaking innovation in this area was provided by Gallant (1981) in his introduction of the semi-nonparametric inference approach, which uses series expansions in infinite dimensional parameter spaces. The idea behind the semi-nonparametric approach is to expand the order of the series expansion, as the sample size increases, until the semi-nonparametric function converges asymptotically to the true function generating the data. Semi-nonparametric functional forms are globally flexible in the sense that the model asymptotically can reach any continuous function. Two globally flexible functional forms in general use are the Fourier flexible functional form, introduced by Gallant (1981), and the Asymptotically Ideal Model (AIM), introduced by Barnett and Jonas (1983) and employed and explained in Barnett and Yue (1988) and Barnett *et al.* (1999).

The problem with the functional approach to monetary aggregation (or aggregation in general) is that the functional aggregate will depend on the specific flexible functional form that is used to approximate the underlying aggregator function, in the same way as statistical monetary aggregates depend on the statistical index number formula used. Moreover, the function must be estimated over specific data sets and re-estimated periodically. This dependence is particularly troublesome to government agencies, and it is exacerbated by the fact that there are many possible flexible functional forms from which to choose. Under these circumstances, government agencies around the world have always viewed aggregation theory as being solely a research tool, and have instead used index number formulas from statistical index number theory that we discussed in the introduction.

In this paper, we take what Barnett (1997) refers to as the 'high road' approach to the demand for money and monetary aggregation. We build on recent advances in microeconometrics and the increase in computational power that allow the estimation of very detailed monetary asset demand systems, without making any restrictive separability assumptions to restrict the dimension of the parameter space. We use two of the most popular, effectively globally regular flexible functional forms to estimate highly disaggregated demand systems, and produce two functional monetary aggregates encompassing the full range of monetary assets in the United States, consistent with neoclassical microeconomic theory and aggregation theory.

3 Flexible Functional Forms

In this paper, we use two flexible functional forms to approximate the underlying unknown indirect utility function, $h(\boldsymbol{p}, y)$. The first is the Minflex Laurent (ML) model, documented in detail in Barnett (1983) and Barnett and Lee (1985), and the second is the normalized quadratic (NQ) expenditure function, introduced by Diewert and Wales (1988).

3.1 The Minflex Laurent Model

The ML model is based on the Laurent series expansion, which is a generalization of the Taylor series expansion. It is also known as the Minflex Laurent generalized Leontief model, as it represents a generalization of the generalized Leontief model, proposed by Diewert (1974), and which is based on a Taylor series expansion.

The ML reciprocal indirect utility function can be written as

$$1/h(\boldsymbol{v}) = c + 2\boldsymbol{\delta}'\sqrt{\boldsymbol{v}} + \sum_{i=1}^{n} d_{ii}\nu_i + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} d_{ij}^2 \nu_i^{\frac{1}{2}} \nu_j^{\frac{1}{2}} - \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} h_{ij}^2 \nu_i^{-\frac{1}{2}} \nu_j^{-\frac{1}{2}} \quad (5)$$

where n denotes the number of goods, ν_i denotes the income normalized price (p_i/y), c is a constant, and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)'$, and d_{ij} and h_{ij} are all parameters.

The share equations of the ML demand system (for $i = 1, \ldots, n$)

$$s_i = \frac{\delta_i \nu_i^{\frac{1}{2}} + d_{ii}\nu_i + \sum_{j=1,j\neq i}^{n} d_{ij}^2 \nu_i^{\frac{1}{2}} \nu_j^{\frac{1}{2}} + \sum_{j=1,j\neq i}^{n} h_{ij}^2 \nu_i^{-\frac{1}{2}} \nu_j^{-\frac{1}{2}}}{\boldsymbol{\delta}'\sqrt{\boldsymbol{v}} + \sum_{i=1}^{n} d_{ii}\nu_i + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} d_{ij}^2 \nu_i^{\frac{1}{2}} \nu_j^{\frac{1}{2}} - \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} h_{ij}^2 \nu_i^{-\frac{1}{2}} \nu_j^{-\frac{1}{2}}}. \quad (6)$$

Note the share equations (6) are homogeneous of degree of zero in the parameters. Therefore, following Barnett and Lee (1985), we impose the normalization

$$\sum_{i=1}^{n} d_{ii} + 2\sum_{i=1}^{n} \delta_i + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} d_{ij}^2 - \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} h_{ij}^2 = 1 \quad (7)$$

and the restrictions

$$d_{ij} = d_{ji}, \quad h_{ij} = h_{ji}, \quad d_{ij}h_{ij} = 0, \quad i \neq j \quad (8)$$

leaving $n^2 + n$ linearly independent parameters to be estimated.

3.2 The Normalized Quadratic Model

The NQ expenditure function is defined as follows

$$C(\boldsymbol{p}, u) = \boldsymbol{\theta}'\boldsymbol{p} + \left(\boldsymbol{b}'\boldsymbol{p} + \frac{1}{2}\frac{\boldsymbol{\pi}'\boldsymbol{B}\boldsymbol{p}}{\boldsymbol{\alpha}'\boldsymbol{p}}\right) u \quad (9)$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)'$, $\boldsymbol{b} = (b_1, \ldots, b_n)'$, and the elements of the $n \times n$ matrix $\boldsymbol{B} \equiv [\beta_{ij}]$ are the unknown parameters to be estimated. Its corresponding indirect utility function is also given by Diewert and Wales (1988) and it is

$$h(\boldsymbol{v}) = \frac{1 - \boldsymbol{\theta}'\boldsymbol{v}}{\boldsymbol{b}'\boldsymbol{v} + \frac{1}{2}(\boldsymbol{\alpha}'\boldsymbol{v})^{-1}\boldsymbol{v}'\boldsymbol{B}\boldsymbol{v}}. \quad (10)$$

To ensure the flexibility and Gorman polar form of the NQ form, we follow Diewert and Wales (1988) and impose the following restrictions

$$\sum_{i=1}^{n} \alpha_i p_i^* = 1, \quad \alpha_i \geq 0 \quad \forall i \quad (11)$$

$$\sum_{i=1}^{n} \theta_i p_i^* = 0 \quad (12)$$

and

$$\sum_{j=1}^{n} \beta_{ij} p_j^* = 0 \quad \forall i \quad \text{and} \quad \beta_{ij} = \beta_{ji}, \quad \forall i, j \tag{13}$$

where $\boldsymbol{p}^* \gg \mathbf{0}_n$ is a reference (or base-period) vector of normalized prices, determined in such a way that $\boldsymbol{p}^* = \mathbf{1}_n$. The non-negative vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)'$ is predetermined as a vector of ones ($\boldsymbol{\alpha} = \mathbf{1}_n$) — see Diewert and Wales (1988) for more details.

The NQ demand system in budget share form is

$$\boldsymbol{s}(\boldsymbol{v}) = \widehat{\boldsymbol{v}}\boldsymbol{\theta} + \widehat{\boldsymbol{v}} \frac{\boldsymbol{b}\boldsymbol{\alpha}^T\boldsymbol{v} + \boldsymbol{B}\boldsymbol{v} - 0.5(\boldsymbol{\alpha}^T\boldsymbol{v})^{-1}\boldsymbol{v}^T\boldsymbol{B}\boldsymbol{v}\boldsymbol{\alpha}}{\boldsymbol{b}^T\boldsymbol{v}\boldsymbol{\alpha}^T\boldsymbol{v} + 0.5\boldsymbol{v}^T\boldsymbol{B}\boldsymbol{v}} \times \left(1 - \boldsymbol{\theta}^T\boldsymbol{v}\right) \tag{14}$$

with $(n^2 + n)/2 + 2n$ linearly independent parameters to be estimated.

4 Demand System Estimation

4.1 Stochastic Specification

In order to estimate demand systems such as (6) and (14), a stochastic version must be specified. It is typically assumed that the observed share in the jth equation deviates from the true share by an additive disturbance term ϵ_j. With the addition of additive errors, each of the share equation systems (6) and (14) can be written in matrix notation as

$$\boldsymbol{s}_t = \boldsymbol{s}(\boldsymbol{v}_t, \boldsymbol{\theta}) + \boldsymbol{\epsilon}_t \tag{15}$$

where $\boldsymbol{s}_t = (s_{1t}, \ldots, s_{nt})'$ is a vector of budget shares with the jth element being $s_j = p_j x_j / y$, $\boldsymbol{v}_t = (v_{1t}, ..., v_{nt})'$ is a vector of expenditure normalized prices with the jth element being $v_j = p_j / y$, $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, ..., \epsilon_{nt})'$ is a vector of classical disturbance terms, and $\boldsymbol{\theta}$ is the parameter vector to be estimated.

4.2 Singularity and Invariance

When estimating $\boldsymbol{\theta}$ in equation (15), a typical assumption about $\boldsymbol{\epsilon}_t$ is homoscedasticity. This assumption requires

$$\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \boldsymbol{H})$$

where $\mathbf{0}$ is an n-dimensional null vector and $\boldsymbol{H}$ is the $n \times n$ covariance matrix. The assumption of a classical disturbance term permits correlation among the disturbances at time t but rules out the possibility of autocorrelated disturbances. Since the demand system satisfies the adding up property, the error covariance matrix $\boldsymbol{H}$ is singular. This introduces a technical problem when the demand system is estimated, since either generalized least squares or maximum likelihood (ML) needs to invert the covariance matrix, $\boldsymbol{H}$. Barten (1969) and McLaren (1990) show that maximum likelihood estimates can be obtained by arbitrarily dropping any good (or, equivalently, equation) in the system (15).

It is to be noted that recently Serletis and Isakin (2017) and Serletis and Xu (2020) relax the homoscedasticity assumption and instead assume that the covariance matrix of the errors,

$\boldsymbol{H}$, is time-varying. In doing so, they consider the VECH and BEKK parameterizations of the variance model, and analytically prove the invariance of the maximum likelihood estimator with respect to the choice of the good deleted from a singular demand system. We are not relaxing the homoscedasticity assumption in this paper, because of the computational problems in the large parameters space that the parameterizations of the variance model introduces.

4.3 Estimation with Autoregressive Disturbances

Autocorrelation in liquid asset demand systems is a common result, potentially caused by institutional constraints which present economic agents from adjusting their asset holdings within one period. In such cases, usually a first-order autoregressive process is assumed such that

$$\boldsymbol{\epsilon}_t = \boldsymbol{\rho}\boldsymbol{\epsilon}_{t-1} + \boldsymbol{\xi}_t$$

where $\boldsymbol{\rho} = [\rho_{ij}]$ is a matrix of unknown parameters and $\boldsymbol{\xi}_t$ is a non-autocorrelated vector disturbance term with constant covariance matrix. In this case, ML estimates of the parameters can be obtained by using a result developed by Berndt and Savin (1975). They showed that if there is a matrix of autocorrelation parameters $\boldsymbol{\rho}$ then adding up implies that the columns of this matrix must add to a common constant. So the maximum likelihood estimation only requires that this constraint be incorporated to achieve invariance to the deleted equation. They also showed that if one assumes no autocorrelation across equations (i.e., $\boldsymbol{\rho}$ is diagonal), then the diagonal elements must all be equal, $\rho_{11} = ... = \rho_{nn} = \rho$. Thus, by writing equation (15) for period $t-1$, multiplying by $\boldsymbol{\rho}$, and subtracting from (15), we can estimate stochastic budget share equations given by

$$\boldsymbol{s}_t = \boldsymbol{s}(\boldsymbol{v}_t, \boldsymbol{\theta}) + \boldsymbol{\rho}\boldsymbol{s}_{t-1} - \boldsymbol{\rho}\boldsymbol{s}(\boldsymbol{v}_{t-1}, \boldsymbol{\theta}) + \boldsymbol{\xi}_t. \tag{16}$$

In this paper, we follow Mochini and Moro (1994) and choose a more parsimonious and flexible specification of the autocorrelation matrix, $\boldsymbol{\rho}$. Mochini and Moro (1994) shows that a singular autcorrelation matrix $\boldsymbol{\rho}$, whose columns add to 0, will not result in any loss of generality. Assuming the autcorrelation matrix $\boldsymbol{\rho}$ is singular, Mochini and Moro (1994) suggests that $\boldsymbol{\rho}$ can be written as

$$\boldsymbol{\rho} = \boldsymbol{\Lambda} - \frac{\boldsymbol{\lambda}\boldsymbol{\lambda}'}{\sum_{z=1}^{n}\lambda_z}$$

where $\boldsymbol{\Lambda}$ is an $n \times n$ matrix and $\boldsymbol{\lambda}$ is an $n \times 1$ vector, defined as

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

Therefore, $\boldsymbol{\rho}$ is modeled to be a singular, symmetric, and dependent on n free parameters, λ_i $(i = 1, ..., n)$, matrix. Let $\boldsymbol{\omega}$ be the $(n-1) \times 1$ vector of share equations after deleting the ith equation, where $i \in [1, n]$. Then the subsystem, which is obtained by deleting the ith equation in (16) is written as

$$\boldsymbol{\omega}_t = \omega\left(\boldsymbol{v}_t, \boldsymbol{\theta}\right) + \boldsymbol{\phi}\boldsymbol{\omega}_{t-1} - \boldsymbol{\phi}\omega\left(\boldsymbol{v}_{t-1}, \boldsymbol{\theta}\right) + \boldsymbol{\zeta}_t$$

where $\boldsymbol{\phi}$ is a matrix whose elements are

$$\phi_{jj} = \lambda_j - \frac{\lambda_j(\lambda_j - \lambda_i)}{\sum_{z=1}^{n} \lambda_z}$$

and

$$\phi_{jk} = -\frac{\lambda_j(\lambda_k - \lambda_i)}{\sum_{z=1}^{n} \lambda_z}, \quad j \neq k.$$

In what follows, we use the Mochini and Moro (1994) specification of the autocorrelation matrix in the estimation of the ML and NQ demand systems with $n = 13$ monetary assets.

4.4 Theoretical Regularity

We also pay explicit attention to the theoretical regularity conditions of positivity, monotonicity, and curvature in our ML and NQ demand systems. As noted by Barnett (2002), without satisfaction of theoretical regularity, "the second-order conditions for optimizing behavior fail, and duality theory fails. The resulting first-order conditions, demand functions, and supply functions become invalid."

The theoretical regularity conditions can be checked as follows [see Barnett and Serletis (2008) for more details]:

- Positivity is checked by direct computation of the estimated indirect utility function $\widehat{h}(\boldsymbol{v})$; it is satisfied, if $\widehat{h}(\boldsymbol{v}) > 0$, for all t.
- Monotonicity is checked by direct computation of the values of the first gradient vector of the estimated indirect utility function; it is satisfied, if $\nabla\widehat{h}(\boldsymbol{v}) < 0$.
- Curvature requires that the Slutsky matrix be negative semidefinite and can be checked by performing a Cholesky factorization of that matrix; it is satisfied, if the Cholesky values are nonpositive. See Lau (1978, Theorem 3.2).

If the curvature conditions of the ML and NQ systems are not satisfied, we impose curvature in order to produce functional monetary aggregates that are consistent with neoclassical microeconomic theory. In particular, in the case of the ML demand system, we follow Barnett (1983) and impose the curvature condition globally by replacing all non-squared parameters by squared parameters. In the case of the NQ system, we follow Diewert and Wales (1988), and impose global concavity by setting $\boldsymbol{B} = -\boldsymbol{K}\boldsymbol{K}'$, where $\boldsymbol{K} = [k_{ij}]$ is a lower triangular matrix. For example, in the case with three goods ($n = 3$), concavity of the NQ expenditure function can be imposed by replacing the elements of $\boldsymbol{B}$ in (14) by the elements of $\boldsymbol{K}$, as follows

$$\beta_{11} = -k_{11}^2$$
$$\beta_{12} = -k_{11}k_{12}$$
$$\beta_{22} = -(k_{12}^2 + k_{22}^2).$$

The other elements of $\boldsymbol{B}$ can be recovered using restriction (13) as follows

$$\begin{aligned} \beta_{13} &= -(\beta_{11} + \beta_{12}) \\ \beta_{23} &= -(\beta_{12} + \beta_{22}) \\ \beta_{33} &= \beta_{11} + 2\beta_{12} + \beta_{22}. \end{aligned}$$

5 Empirical Results

5.1 Data

We follow Jadidzadeh and Serletis (2019) and use the monthly time series data on monetary asset quantities and their user costs recently produced for the United States by Barnett *et al.* (2013), and maintained within the CFS program Advances in Monetary and Financial Measurement (AMFM). Although the CFS data goes back to 1967:1, we begin our demand analysis in 1974:6, because some key assets (including money-market funds) were not introduced into the U.S. financial system until the mid-1970s. Thus, our sample period is from 1974:6 to 2018:5 (a total of 526 observations).

In particular, we model the demand for the 13 monetary assets listed in Table 1 and included in the broadest CFS Divisia M4 monetary aggregate. As we require real per capita asset quantities for the empirical work, we divide each quantity series by the CPI (all items) and total population. We also add 0.01 to all the real user cost series obtained from the CFS in order to deal with some zero real user cost observations in the original data. We multiply the modified real user costs by the CPI to get nominal user costs. For a detailed discussion of the data and the methodology for the calculation of user costs, see Barnett *et al.* (2013) and http://www.centerforfinancialstability.org.

5.2 Maximum Likelihood Estimation

We estimate the models using the full information maximum likelihood procedure in RATS 9.2 (32). Both models are estimated with the curvature conditions imposed, as discussed in the previous section, and there are no induced positivity and monotonicity violations, thus obtaining parameter estimates that are consistent with all three theoretical regularity conditions at every point in the sample. We do not report the 182 parameters with the ML demand system and the 117 parameters with the NQ demand system (and their standard errors), but note that almost all of them are statistically significant at conventional significance levels. We report the implied autocorrelation matrix $\boldsymbol{\rho}$ by the estimated $\boldsymbol{\lambda}$ for each of the ML and NQ models in Table 2. The parameters on the diagonal of $\boldsymbol{\rho}$ suggest that the error terms have a high degree of autocorrelation.

5.3 The ML, NQ, and Divisia Monetary Aggregates

In this paper, we are not interested in the estimated income (expenditure) elasticities, own- and cross-price elasticities, and Allen and Morishima elasticities of substitution. Instead, we use the maximum likelihood parameter estimates to replace the unknown parameters of the indirect utility function, $h(\boldsymbol{p}, y)$, in each of (5) and (10) by their estimates to obtain the ML and NQ functional monetary aggregates, respectively. We then multiply each of the real per capita monetary aggregates by total population and the CPI to convert them to nominal terms and for the overall economy. We also normalize each of the ML and NQ monetary aggregates so that their first observation equals 100.

The logged ML and NQ functional monetary aggregates are plotted in Figures 1 and 2, respectively, together with their growth rates (on the y_2 axis); the shaded areas indicate NBER recessions. For comparison purposes, in Figure 3 we present the ML and NQ aggregates together with the Fed's Sum M2 aggregate (on the y_2 axis) and the CFS Divisia M3 and Divisia M4 aggregates; the Divisia M3 and M4 series are also normalized so that the June 1974 observation is 100. Finally, in Figure 4 we present the annualized growth rates for all five monetary aggregates. According to these figures, the dynamics of the ML and NQ functional monetary aggregates are quite similar.

As can be seen in Figure 4, the dynamics of the Fed's Sum M2 aggregate are very different from the dynamics of the ML, NQ, and Divisia monetary aggregates, confirming our earlier claim that the Fed's simple-sum monetary aggregates are seriously flawed. They impose assumptions on the substitutability between monetary assets that are extreme and counterfactual, and have repeatedly been shown in empirical work to yield misleading conclusions about the role that money plays in the economy. The ML and NQ monetary aggregates behave similarly to the Divisia aggregates. The fact that the functional aggregates track closely with the Divisia aggregates is an important specification test, confirming that the ML and NQ flexible functional forms that we used to approximate the unknown underlying aggregator function are state of the art.

However, there are some differences between the ML, NQ, and Divisia monetary aggregates at a small number of critical points in U.S. monetary history. In particular, the ML and NQ aggregates show continuing rapid growth throughout the late 1970s; the Divisia aggregates, by contrast, show marked deceleration well before Paul Volcker's arrival at the Fed. Also, growth in the functional aggregates moves sharply higher at the end of the "monetarist experiment" in 1983 and 1984. The functional aggregates also exhibit faster growth around 1995 than the Divisia aggregates do. Moreover, although growth in the ML, NQ, and Divisia monetary aggregates fell sharply into negative territory in the aftermath of the financial crisis and Great Recession of 2007-2009, suggesting a monetary explanation for the low inflation and sluggish growth observed over that period, the ML aggregate shows a more dramatic collapse; at the same time, the ML aggregate also shows exceptionally rapid growth during the Great Recession. Finally, unlike the Divisia aggregates, both functional aggregates show extremely rapid growth over the past couple of years.

Regarding these small differences between the ML, NQ, and Divisia monetary aggregates, it is to be noted that the Divisia index is consistent with microeconomic theory and aggregation theory, so long as economic agents are optimizing under perfect certainty and the time series of prices and quantities are generated by a homothetic function. Under those conditions, as a measure of the observed optimized service flow, the (chained) Divisia index tracks the optimized value of the unknown aggregator function up to a very small third order error in the log changes, and is superior to any estimated aggregator function having a finite number of fixed parameters. However, as Samuelson and Swamy (1974, p. 592) conclude, "empirical experience is abundant that the Santa Claus hypothesis of homotheticity in tastes and in technical change is quite unrealistic. Therefore, we must not be bemused by the undoubted elegances and richness of the homothetic theory. ... we must accept the sad facts of life, and be grateful for the more complicated procedures economic theory devises."

With this in mind, in what follows we provide a comprehensive comparison between the ML and NQ functional monetary aggregates and the Fed's simple-sum M2 monetary aggregate and the two broad statistical monetary aggregates from the Center for Financial Stability, Divisia M3 and Divisia M4.

6 Business Cycle Analysis

In this section we use two dynamic general equilibrium monetary business models. The first model is from Ireland (2004) and the second model is from Andrés *et al.* (2006). Both of these articles find a minimal role of money in shaping business cycles.

6.1 The Ireland (2004) Model

This is a small dynamic general equilibrium monetary business model that incorporates money balances in the IS and Phillips curve specifications of the Rotemberg (1982) and Rotemberg and Woodford (1997) new Keynesian model. It is assumed that monopolistically competitive firms face a quadratic cost of nominal price adjustment so that prices are sticky. It is also assumed that a representative consumer maximizes expected utility by choosing consumption, labor supply, and real money balances. Notably, the model allows real money balances to enter the forward-looking IS and Phillips curve specifications. The model is closed by adding a Taylor-type monetary policy rule. See Ireland (2004) for more details.

The first-order conditions describing the optimizing behavior of the representative household and intermediate goods-producing firm of the Ireland (2004) model can be approximated by

$$\begin{aligned}
\hat{y} &= E_t\hat{y}_{t+1} - \omega_1(\hat{r}_t - E_t\hat{\pi}_{t+1}) \\
&\quad + \omega_2(\hat{m}_t - \hat{e}_t - E_t\hat{m}_{t+1} + E_t\hat{e}_{t-1}) + \omega_1(\hat{a}_t - E_t\hat{a}_{t+1}) &(17)\\
\hat{m}_t &= \gamma_1\hat{y}_t - \gamma_2\hat{r}_t + \gamma_3\hat{e}_t &(18)\\
\hat{\pi}_t &= \beta E_t\hat{\pi}_{t+1} + \psi\left[\frac{1}{\omega_1}\hat{y}_t - \frac{\omega_2}{\omega_1}(\hat{m}_t - \hat{e}_t) - \hat{z}_t\right] &(19)\\
\log\hat{a}_t &= \rho_a \log\hat{a}_{t-1} + \epsilon_{at} &(20)\\
\log\hat{e}_t &= \rho_e \log\hat{e}_{t-1} + \epsilon_{et} &(21)\\
\log\hat{z}_t &= \rho_z \log\hat{z}_{t-1} + \epsilon_{zt} &(22)
\end{aligned}$$

where y_t is real output, m_t is real money balances, r_t is the gross nominal interest rate, π_t is the gross inflation rate, and the hatted variables are the percentage (logarithmic) deviations from their steady-state values. β is the discount factor, a_t is a preference shock, e_t a money demand shock, and z_t a technology shock. It is assumed that $\epsilon_{at} \sim N(0, \sigma_a^2)$, $\epsilon_{et} \sim N(0, \sigma_e^2)$, and $\epsilon_{zt} \sim N(0, \sigma_z^2)$. Moreover, the following interest rate rule is added to close the model

$$\hat{r}_t = p_r\hat{r}_{t-1} + p_y\hat{y}_{t-1} + p_\pi\hat{\pi}_{t-1} + \epsilon_{rt} \tag{23}$$

where $\epsilon_{rt} \sim N(0, \sigma_r^2)$. There are two restrictions imposed on the parameters

$$\gamma_1 = \left(r - 1 + \frac{yr\omega_2}{m}\right)\frac{\gamma_2}{\omega_1}$$

and

$$\gamma_3 = 1 - (r-1)\gamma_2$$

where r, y, π, and m are the steady-state values of the gross nominal interest, real output, the gross inflation rate, and real money balances.

Ireland (2004) estimates (17)-(23) using U.S. quarterly data from 1980:1 to 2001:3. He assumes that $\omega_1 = 1$ and $\psi = 0.1$, and calculates real per capita money balances by dividing the Fed's simple-sum M2 monetary aggregate by the product of the GDP deflator and population. The role of money in this model is captured by ω_2 in equation (18) which is the forward-looking IS curve. This parameter also determines the impact of money on inflation in equation (19) which is the forward-looking Phillips curve. Ireland (2004) estimates the model using maximum likelihood estimation and finds that ω_2 is essentially zero, suggesting that real balances fail to enter into the IS and Phillips curve equations that govern the dynamics of output and inflation.

While Ireland (2004) uses quarterly data from 1980:1 to 2001:3, the focus here on the period from 1974:3 to 2018:1 allows for the use of more recent data over a period that includes the global financial crisis and great recession and its aftermath. We obtain real per capita GDP by dividing real GDP by the civilian noninstitutional population, we measure the inflation rate using the GDP deflator, and use the federal funds rate and the shadow federal funds rate from Wu and Xia (2016) for the interest rate. All the series are from the Federal Reserve Bank of St. Louis FRED data base. Following Ireland (2004), we use a linear trend to detrend the logs of per capita output and per capita real money balances.

We estimate (17)-(23), also imposing the two constraints, $\omega_1 = 1$ and $\psi = 0.1$, using our quarterly ML and NQ functional monetary aggregates, which are already in real per capita terms, since they are obtained from the estimation a demand system based on the representative economic agent. For comparison purposes, we also report results with the Fed's Sum M2 aggregate and the CFS Divisia M3 and Divisia M4 aggregates. The maximum likelihood parameter estimates are reported in Table 3. As can be seen, the results with the Sum M2 monetary aggregate are almost the same as those reported by Ireland (2004); $\hat{\omega}_2$ is very small ($\hat{\omega}_2 = 0.0006$) and statistically insignificant (the standard error is 0.0005), providing evidence for the version of the model in which real money balances are completely absent from the IS and Phillips curve specifications. The results with the Divisia M3 and Divisia M4 aggregates indicate that although $\hat{\omega}_2$ is very small ($\hat{\omega}_2 = 0.0002$ with a standard error of 0.0000 with Divisia M3 and $\hat{\omega}_2 = 0.0001$ with a standard error of 0.0001 with Divisia M4), it is significantly different from zero, suggesting that movements in the broad Divisia monetary aggregates have an effect on the dynamics of output and inflation.

However, our maximum likelihood estimates with the functional monetary aggregates, ML and NQ, tell a very different story compared with Ireland (2004). We find that real money balances could matter, since $\hat{\omega}_2$ is relatively sizable with both the ML and NQ monetary aggregates ($\hat{\omega}_2 = 0.0084$ with the ML aggregate and $\hat{\omega}_2 = 0.0059$ with the NQ aggregate), even though they also have larger standard errors. This suggests that the presence of real money balances in the IS and Phillips curve specifications should be important in explaining fluctuations in the data. We also find that the income elasticity, $\hat{\gamma}_1$, and interest semi-elasticity of money demand, $\hat{\gamma}_2$, are small. Moreover, we find that the money shock, $\hat{e}_t$, drives the dynamics of real money balances, $\hat{m}_t$, significantly since $\hat{\gamma}_3$ is quite large. The estimates of the other parameters are consistent with those obtained by Ireland (2004).

6.2 The Andrés *et al.* (2006) Model

The Andrés *et al.* (2006) model is similar to the Ireland (2004) model. It consists of a representative household, a continuum of producing firms, and a monetary authority. It allows for habits in consumption and for non-separability among consumption and real balances in preferences, as emphasized by Fuhrer (2000) and Christiano *et al.* (2005), making it possible to test the relevance of a direct effect of money balances on supply and demand decisions. It also assumes monopolistically competitive markets where a representative firm sells its output and sets nominal prices on a staggered basis, as in Calvo (1983). The model is closed assuming an augmented Taylor-type rule that the central bank uses to set the nominal interest rate in response not only to the interest rate in the previous period and to the inflation and output gaps, but also to money growth. Regarding the assumption that money growth enters the Taylor rule, it should be noted that the original draft of Ireland (2004), still available at https://www.nber.org/papers/w8115.pdf, also made this same assumption, and also found a statistically significant response of the policy rate to changes in money growth.

The log-linearized first-order conditions describing the optimizing behavior of the representative household and intermediate goods-producing firm of the Andrés *et al.* (2006)

model are

$$
\begin{aligned}
\hat{y}_t &= \frac{\phi_1}{\phi_1+\phi_2}\hat{y}_{t-1} + \frac{\beta\phi_1+\phi_2}{\phi_1+\phi_2}E_t\hat{y}_{t+1} - \frac{1}{\phi_1+\phi_2}\left(\hat{r}_t - E_t\pi_{t+1}\right) - \frac{\beta\phi_1}{\phi_1+\phi_2}E_t\hat{y}_{t+2} \\
&\quad + \frac{\psi_2}{\psi_1}\frac{1}{1-\beta h}\frac{1}{\phi_1+\phi_2}(\hat{m}_t - \hat{e}_t) - \frac{\psi_2}{\psi_1}\frac{1}{1-\beta h}\frac{1+\beta h}{\phi_1+\phi_2}E_t(\hat{m}_{t+1} - \hat{e}_{t+1}) \\
&\quad + \frac{\psi_2}{\psi_1}\frac{1}{\beta h}\frac{1+\beta h}{\phi_1+\phi_2}E_t(\hat{m}_{t+2} - \hat{e}_{t+2}) + \frac{1-\beta h\rho_a}{1-\beta h}\frac{1-\rho_a}{\phi_1+\phi_2}\hat{a}_t \qquad (24) \\
\hat{m}_t &= \gamma_1\hat{y}_t - \gamma_2\hat{r}_t + \left[\gamma_2(r-1)(h\phi_2-\phi_1) - h\gamma_1\right]\hat{y}_{t-1} \\
&\quad - [\gamma_2(r-1)\beta\phi_1]E_t\hat{y}_{t+1} + \frac{\psi_2}{\psi_1}\frac{(r-1)\beta h\gamma_2}{1-\beta h}E_t\hat{m}_{t+1} \\
&\quad - \frac{(r-1)\beta h(1-\rho_a)}{1-\beta h}\gamma_2\hat{a}_t + \left[1-(r-1)\gamma_2\left(\frac{\psi_2}{\psi_1}\frac{\beta h\rho_e}{1-\beta h}+1\right)\right]\hat{e}_t \qquad (25) \\
\hat{\mu}_t &= \hat{m}_t - \hat{m}_{t-1} + \hat{\pi}_t \qquad (26) \\
\hat{\pi} &= \gamma_f E_t\hat{\pi}_{t+1} + \gamma_b\hat{\pi}_{t-1} + \lambda\hat{mc}_t \qquad (27) \\
\hat{mc}_t &= (\chi+\phi_2)\hat{y}_t - \phi_1\hat{y}_{t-1} - \beta\phi_1E_t\hat{y}_{t+1} - \frac{\psi_2}{\psi_1}\frac{1}{1-\beta h}(\hat{m}_t - \hat{e}_t) \\
&\quad + \frac{\psi_2}{\psi_1}\frac{\beta h}{1-\beta h}E_t(\hat{m}_{t+1} - \hat{e}_{t+1}) - \frac{\beta h(1-\rho_a)}{1-\beta h}\hat{a}_t - (1+\chi)\hat{z}_t \qquad (28) \\
\log\hat{a}_t &= \rho_a\log\hat{a}_{t-1} + \epsilon_{at} \qquad (29) \\
\log\hat{e}_t &= \rho_e\log\hat{e}_{t-1} + \epsilon_{et} \qquad (30) \\
\log\hat{z}_t &= \rho_z\log\hat{z}_{t-1} + \epsilon_{zt}, \qquad (31)
\end{aligned}
$$

where h measures the habits in consumption.

The monetary policy rule considered by Andrés *et al.* (2006) is

$$\hat{r}_t = \rho_r\hat{r}_{t-1} + (1-\rho_r)\rho_y\hat{y}_t + (1-\rho_r)\rho_\pi\hat{\pi}_t + (1-\rho_r)\rho_\mu\hat{\mu}_t + \epsilon r_t \qquad (32)$$

where $\epsilon_{rt} \sim N(0,\sigma_r^2)$. This policy rule is different from the one used by Ireland (2004), since the monetary policymaker responds to the deviation of the money growth rate from its steady-state value, $\hat{\mu}_t$. We can certainly restrict $\rho_\mu = 0$ for a conventional interest rule used in the literature for the U.S. economy. However, we keep it in the estimation since we will be able to test if $\rho_\mu = 0$ is supported by the data. Moreover, Belongia and Ireland (2019) report evidence that supports the inclusion of money in Taylor-type interest rate policy rules. In this model, ψ_2 is the key parameter, showing the impact of money balances on the economy, and Andrés *et al.* (2006) find that ψ_2 is zero using euro zone data and the simple-sum M3 monetary aggregate.

We estimate equations (24)-(32) using the same data we used to estimate the Ireland (2004) model in the previous section. In doing so, we calibrate β to 0.99 which matches the value used in the literature. We follow Andrés *et al.* (2006) and also impose the constraints $\gamma_f = \beta$ and $\gamma_b = 1 - \gamma_f$, since γ_b is close to zero when the model is estimated without any restrictions. We present the maximum likelihood parameter estimates in Table 4, in the same fashion as those in Table 3 with the Ireland (2004) model.

As can be seen, the null hypothesis that $\hat{\psi}_2 = 0$ is rejected with all the monetary aggregates, except for Sum M2, with the margins of rejection being higher with the ML and

NQ aggregates than with the Divisia M3 and Divisia M4 aggregates. The ψ_2 parameter governs the separability of the utility function on real balances, and in contrast to Andrés *et al.* (2006) and also Ireland (2004) we find that the real balances effect in the dynamics of output and inflation is significantly different from zero. Regarding the ρ_μ parameter, we find that it is statistically significant and different from zero with all the monetary aggregates, indicating a significant response of the interest rate to the money growth rate. As Andrés *et al.* (2006, p. 467) put it, "this effect is less common in the literature and opens up a channel of influence of money balances that may be potentially important."

As regards the other parameters of interest, as in Andrés *et al.* (2006) we find evidence of habit formation, as the h parameter is high and statistically significant. The elasticity of money demand with respect to output, γ_1, is close to 1 with the ML and NQ aggregates, around 0.7 with the Divisia M3 and Divisia M4 aggregates, and close to zero with the Sum M2 aggregate. Finally, the elasticity of money demand with respect to the interest rate, γ_2, is close to zero with the ML and NQ aggregates and the Divisia M3 and Divisia M4 aggregates, but it is above 1 with the Sum M2 aggregate, very much in line with the value obtained by Ireland (2004).

7 Granger Causality

In this section we test for Granger causality from the ML and NQ functional monetary aggregates to industrial production. We also provide a comparison with the Fed's simple-sum M2 monetary aggregate and the CFS Divisia M3 and Divisia M4 monetary aggregates. We carry out the investigation in the context of the standard VAR approach — see Stock and Watson (1989) , Bernanke and Blinder (1992), Swanson (1998), Belongia and Ireland (2015), and Dery and Serletis (2020), among others. However, we also use an alternative approach that addresses the nonlinearity in the time series, since (as it has already been noted in the literature) the money-output relationship may not be stable over the business cycle.

7.1 The Conventional Approach

We assume that the relevant information is contained in the present and past values of the variables and use the following trivariate autoregressive representation

$$\Delta y_t = \alpha + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \sum_{j=1}^{q} \theta_j \Delta m_{t-j} + \sum_{k=1}^{r} \lambda_k \pi_{t-k} + e_t$$

where Δy_t is the first difference of logged output, Δm_t the first difference of logged money, and π_t is the inflation rate (a conditioning variable). We test for Granger causality in the context of a flexible lag structure optimally chosen by the Akaike information criterion (AIC) after letting each of p, q, and r take values from 1 to 12. Also, we test for Granger causality over the full sample, from 1974 to 2018, and three subsamples: (a) the period of the Great Inflation through the Volcker disinflation, 1974-1985, (b) the period of the Great Moderation, 1986-1999, and (c) the period of the crisis, Great Recession, and zero nominal interest rates, 2000-present.

We report the Granger causality test results for each monetary aggregate and for each sample period in Table 5. Each entry in the table represents the marginal significance level of the test statistic testing the null hypothesis that all lags of the monetary growth rate, Δm_t, can be excluded from the regression; that is $\theta_j = 0, \; \forall \; j$. Therefore, smaller p-values indicate a stronger role for money. As can be seen in Table 5, the Sum M2 monetary aggregate has no predictive power at all for economic growth, both over the full sample and the subsamples, confirming that the simple-sum aggregates deliver different and misleading information about monetary policy and the effects it is having on the economy. The Divisia M3 and Divisia M4 aggregates perform well over the full sample and the first subsample, from 1974 to 1985, but their predictive power diminishes in data from the post-1985 period. The ML and NQ functional monetary aggregates Granger cause real output in all cases and their predictive power does not diminish in the post-1985 period, except for the ML aggregate over the third sabsample, from 2000 to 2018. Our conclusion from these tests is that the functional monetary aggregates are more informative for predicting real economic activity.

7.2 Markov Switching Granger Causality

In this section we use the Psaradakis *et al.* (2005) Markov-switching bivariate VAR model (in logged differences) with time-varying parameters, (with parameter time-variation directly) reflecting changes in the causal relation between money and output.[1] In doing so, we treat changes in causality as random events, governed by an unobserved variable which follows a homogeneous Markov chain with four states, governing the money-output relationship over time. That is, we substitute the notion of permanent causality with a notion of temporary causality (causality holding during some periods but not in others).

In particular, following Psaradakis *et al.* (2005), we use the following Markov switching bivariate VAR model

$$\begin{pmatrix} \Delta y_t \\ \Delta m_t \end{pmatrix} = \boldsymbol{C}_{s_t} + \sum_{i=1}^{q} \boldsymbol{A}_{i,s_t} \begin{pmatrix} \Delta y_{t-i} \\ \Delta m_{t-i} \end{pmatrix} + \sum_{i=1}^{r} \boldsymbol{B}_{i,s_t} \pi_{t-i} + \boldsymbol{u}_{t,s_t}, \quad \boldsymbol{u}_{t,s_t} \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_{s_t})$$

where Δy_t is the first difference of logged output, Δm_t the first difference of logged money, π_t the inflation rate (a 'conditioning' variable), and s_t is an unobserved variable which follows a homogeneous Markov chain with four states. The transition between the states is controlled by the transition matrix

$$\boldsymbol{\Pi} = \begin{pmatrix} p_{11} & \cdots & p_{14} \\ \vdots & \ddots & \vdots \\ p_{41} & \cdots & p_{44} \end{pmatrix}$$

where $p_{ji} = P[z_t = j \,|z_{t-1} = i]$, $i, j = 1, ..., m$, and $p_{ji} = 1 - \sum_{k \neq j} p_{ki}$ is the probability of state j in period t given that the system was in state i in period $t - 1$.

[1] Rothman *et al.* (2001) also take a similar approach to the investigation of the nonlinear causality relationship between money and output, but in the context of a smooth-transition model.

The parameters in the $\boldsymbol{C}_{s_t}$, $\boldsymbol{A}_{i,s_t}$, and $\boldsymbol{B}_{i,s_t}$ matrices change with changes in s_t. We impose the following restrictions to help us identify the money-output relationship over time

$$\begin{aligned}
\boldsymbol{A}_{i,s_t} &= \begin{bmatrix} a_{i,11,s_t=1} & a_{i,12,s_t=1} \\ a_{i,21,s_t=1} & a_{i,22,s_t=1} \end{bmatrix}, \quad \text{if } s_t = 1 \\
\boldsymbol{A}_{i,s_t} &= \begin{bmatrix} a_{i,11,s_t=2} & 0 \\ a_{i,21,s_t=2} & a_{i,22,s_t=2} \end{bmatrix}, \quad \text{if } s_t = 2 \\
\boldsymbol{A}_{i,s_t} &= \begin{bmatrix} a_{i,11,s_t=3} & a_{i,12,s_t=3} \\ 0 & a_{i,22,s_t=3} \end{bmatrix}, \quad \text{if } s_t = 3 \\
\boldsymbol{A}_{i,s_t} &= \begin{bmatrix} a_{i,11,s_t=4} & 0 \\ 0 & a_{i,22,s_t=4} \end{bmatrix}, \quad \text{if } s_t = 4.
\end{aligned}$$

That is, the specification of the model allows for four alternative states of nature. In the first state, money is Granger-causal for output ($a_{i,12,s_t=1} \neq 0$) and output is Granger-causal for money ($a_{i,21,s_t=1} \neq 0$). In the second state, money is not Granger-causal for output ($a_{i,12,s_t=2} = 0$), but output is Granger-causal for money ($a_{i,21,s_t=2} \neq 0$). In the third state, money is Granger-causal for output ($a_{i,12,s_t=3} \neq 0$) but output is not Granger-causal for money ($a_{i,21,s_t=3} = 0$). Finally, in the fourth state, there is no Granger causality between money and output ($a_{i,12,s_t=4} = 0$ and $a_{i,21,s_t=4} = 0$).

We use monthly data for the United States, over the period from June 1974 to May 2018, and estimate the model following Hamilton (1994). We use the industrial production index as a proxy for real output and for the money supply we use the ML and NQ functional monetary aggregates (after we multiply by population and the consumer price index to get the aggregate quantity of money) as well as the Fed's Sum M2 aggregate and the CFS Divisia M3 and Divisia M4 aggregates, for comparison purposes. We choose the VAR lag length specification based on the AIC criterion, and calculate the filtered probabilities that money causes output from the consolidated state where money causes output

$$p(s_t = 1|\Omega_t) + p(s_t = 3|\Omega_t) = p(\text{money is Granger causal for output } |\Omega_t)$$

where Ω_t is the information set at time period t.

We report the filtered probabilities that money causes output in Figures 5-9 for the ML and NQ functional monetary aggregates, the Sum M2 aggregate, and the Divisia M3 and Divisia M4 aggregates, respectively. As can be seen in the figures, the switching between the two consolidated states is generally observed during economic contractions such as the 2007-2009 global financial crisis. Moreover, Figures 5 and 6 show that the functional monetary aggregates (and to a larger extent the NQ monetary aggregate) cause real output most of the time since the 1980s. Finally, based on this methodology, the Fed's Sum M2 aggregate and the CFS Divisia M3 and Divisia M4 aggregates do not seem to be good predictors of the dynamics of output most of the time.

We conclude that the functional monetary aggregates are better for predicting real economic activity, but this may not be the case for other uses of monetary aggregates.

8 Conclusion

As Belongia (1996, p. 1082) put it, "simple-sum monetary aggregates can be constructed in a relatively straightforward manner, but they have no basis in either economic or index number theory. Divisia monetary aggregates on the other hand, require more care in construction but are solidly based in theory." The approach in this paper is based on the use of neoclassical microeconomic and aggregation theory to produce functional monetary aggregates, encompassing the full range of monetary assets. This approach requires more care, because it involves the estimation of highly disaggregated monetary asset demand systems, without making any restrictive separability assumptions to restrict the dimension of the parameter space. This was not possible a few years ago, which explains why earlier work in this area has generally been carried out in the context of small, highly aggregated monetary asset demand systems, and the computation of monetary quantity indexes relied exclusively on statistical index number theory.

We build on recent advances in microeconometrics and the increase in computational power that allow the estimation of very detailed demand systems. We use two of the most popular, effectively globally regular flexible functional forms, the minflex Laurent and the normalized quadratic. We address the issue of optimal monetary aggregation (by including the full range of monetary assets), as well as the problems of dimensionality and nonlinearity, estimating very detailed demand systems, treating the curvature property as a maintained hypothesis. Our broad ML and NQ functional monetary aggregates are consistent with neoclassical microeconomic theory and aggregation theory. Our detailed statistical analysis favors the functional monetary aggregates over the statistical monetary aggregates (the Fed's Sum M2 and the CFS Divisia M3 and Divisia M4 monetary aggregates).

We believe that our approach will prove useful in computing monetary quantity indexes, understanding the effects of potential monetary policy actions, and restoring a meaningful role for money within dynamic stochastic general equilibrium frameworks. However, we would also like to reiterate an important point made earlier. In particular, a potential advantage of the Divisia aggregates over the functional aggregates is that the index numbers can be constructed in real time and would not have to be revised as the estimated parameters of the ML and NQ aggregates change over time. In future work, we plan to explore the statistical stability of the ML and NQ parameter estimates. If these parameters are found to be largely stable over time, that would dispel one potential concern about the functional approach to monetary aggregation.

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Table 1. Components and monetary aggregates in the United States

		Monetary Aggregates									
		Federal Reserve		Center for Financial Stability, Divisia							
Mnemonic	Liquid asset	Sum M1	Sum M2	M1	M2M	MZM	M2	ALL	M3	M4-	M4
x_1	Transaction balances	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
	Currency										
	Travelers' checks										
	Demand deposits										
x_2	OCDs at commercial banks	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
x_3	OCDs at thrifts institutions	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
x_4	Saving deposits at banks including MMDAs		$\surd$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
x_5	Saving deposits at thrifts including MMDAs		$\surd$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
x_6	Retail money-market funds		$\surd$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
x_7	Small time deposits at commercial banks		$\surd$				$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
x_8	Small time deposits at thrift institutions		$\surd$				$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
x_9	Institutional money-market funds					$\surd$		$\surd$	$\surd$	$\surd$	$\surd$
x_{10}	Large time deposits								$\surd$	$\surd$	$\surd$
x_{11}	Repurchase Agreements								$\surd$	$\surd$	$\surd$
x_{12}	Commercial paper									$\surd$	$\surd$
x_{13}	T-bills										$\surd$

Table 2. Symmetric autocorrelation matrix

A. ML demand system

$\hat{\boldsymbol{\rho}} =$

0.9171												
−0.0760	0.9166											
−0.0765	−0.0765	0.9223										
−0.0768	−0.0768	−0.0773	0.9258									
−0.0760	−0.0760	−0.0765	−0.0768	0.9170								
−0.0760	−0.0760	−0.0765	−0.0768	−0.0760	0.9167							
−0.0768	−0.0768	−0.0773	−0.0776	−0.0768	−0.0768	0.9258						
−0.0767	−0.0767	−0.0772	−0.0775	−0.0767	−0.0767	−0.0775	0.9247					
−0.0769	−0.0769	−0.0774	−0.0777	−0.0769	−0.0769	−0.0777	−0.0776	0.9269				
−0.0764	−0.0764	−0.0769	−0.0772	−0.0764	−0.0764	−0.0772	−0.0771	−0.0773	0.9211			
−0.0761	−0.0761	−0.0766	−0.0769	−0.0761	−0.0761	−0.0769	−0.0768	−0.0770	−0.0765	0.9179		
−0.0763	−0.0763	−0.0768	−0.0771	−0.0763	−0.0763	−0.0771	−0.0770	−0.0772	−0.0767	−0.0764	0.9203	
−0.0764	−0.0763	−0.0768	−0.0772	−0.0764	−0.0763	−0.0772	−0.0771	−0.0773	−0.0767	−0.0764	−0.0767	0.9207

B. NQ demand system

$\hat{\boldsymbol{\rho}} =$

0.9172												
−0.0768	0.9279											
−0.0762	−0.0772	0.9215										
−0.0767	−0.0777	−0.0771	0.9269									
−0.0761	−0.0771	−0.0765	−0.0770	0.9200								
−0.0766	−0.0755	−0.0769	−0.0774	−0.0768	0.9253							
−0.0767	−0.0777	−0.0771	−0.0776	−0.0770	−0.0775	0.9274						
−0.0763	−0.0773	−0.0767	−0.0772	−0.0766	−0.0770	−0.0772	0.9226					
−0.0767	−0.0777	−0.0771	−0.0776	−0.0770	−0.0775	−0.0777	−0.0772	0.9273				
−0.0765	−0.0775	−0.0769	−0.0774	−0.0767	−0.0772	−0.0774	−0.0770	−0.0774	0.9244			
−0.0763	−0.0772	−0.0766	−0.0771	−0.0765	−0.0770	−0.0772	−0.0768	−0.0772	−0.0769	0.9220		
−0.0763	−0.0773	−0.0767	−0.0772	−0.766	−0.0771	−0.0772	−0.0768	−0.0772	−0.0770	−0.0768	0.9227	
−0.0760	−0.0770	−0.0764	−0.0769	−0.0762	−0.0767	−0.0769	−0.0764	−0.0769	−0.0766	−0.0764	−0.0765	0.9191

Figure 1. The Minflex Laurent monetary aggregate and its growth rate

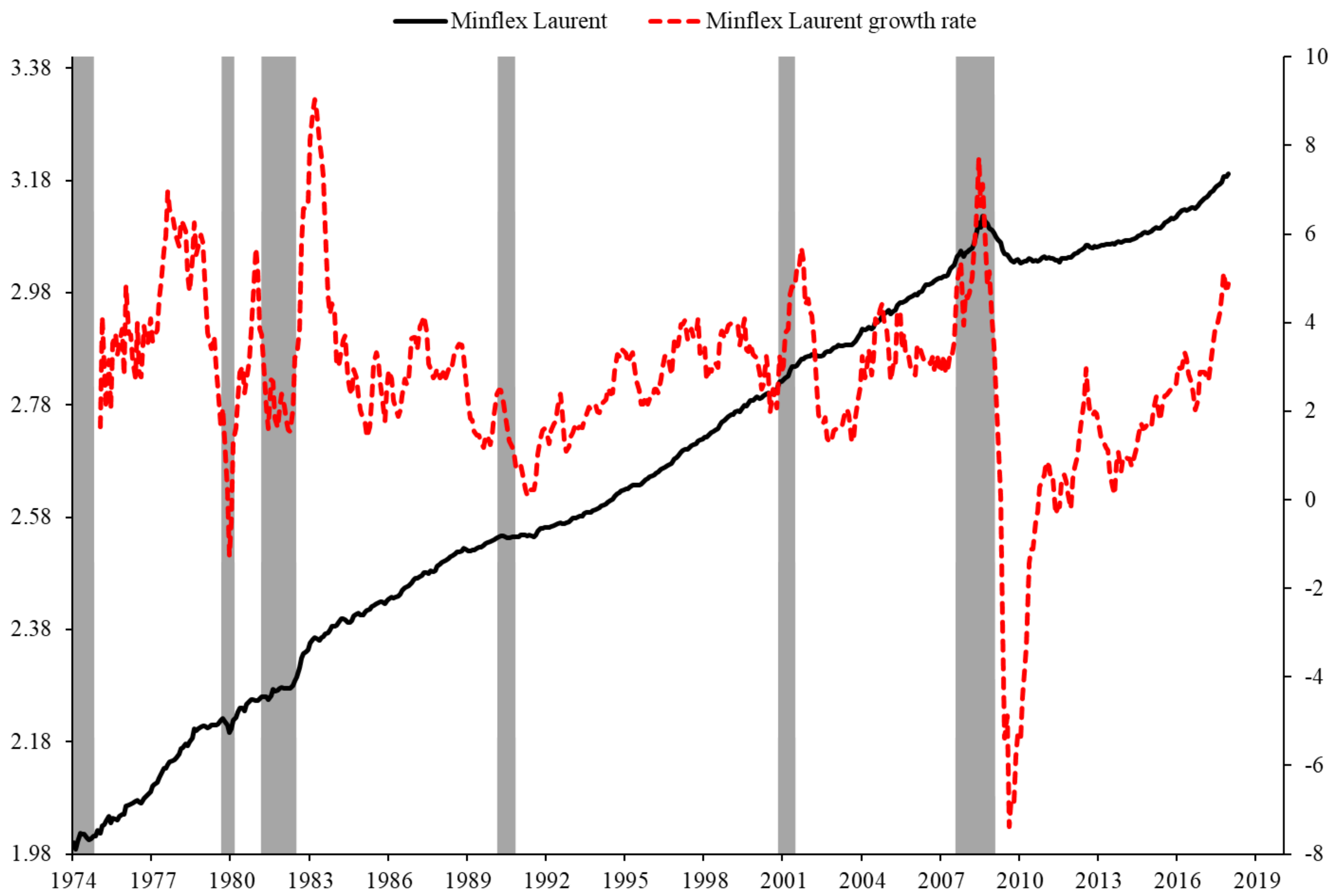


Figure 2. The Normalized Quadratic monetary aggregate and its growth rate

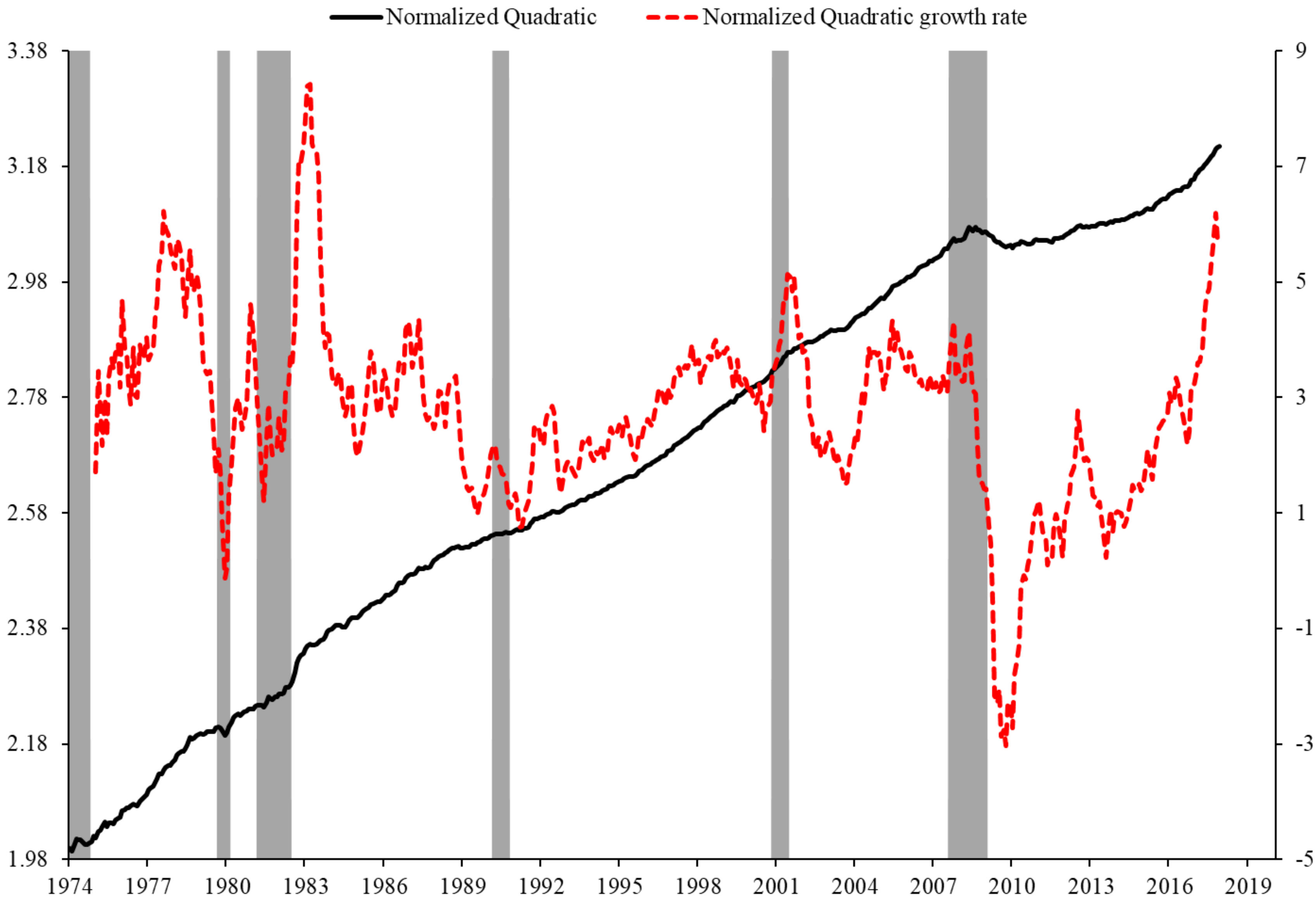


Figure 3. Functional and statistical monetary aggregates (in log levels)

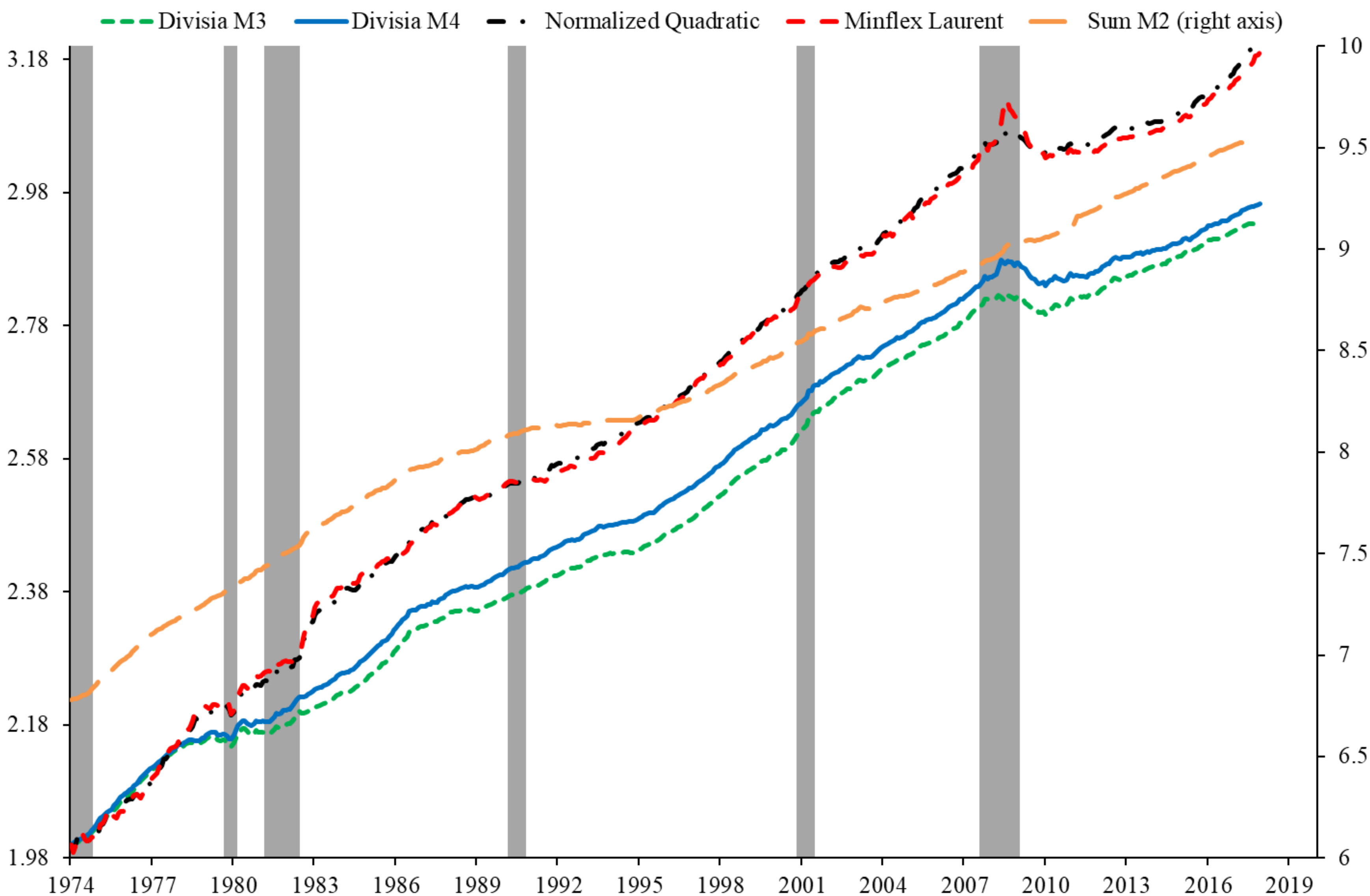


Figure 4. Year-over-year percentage growth rates of functional and statistical aggregates

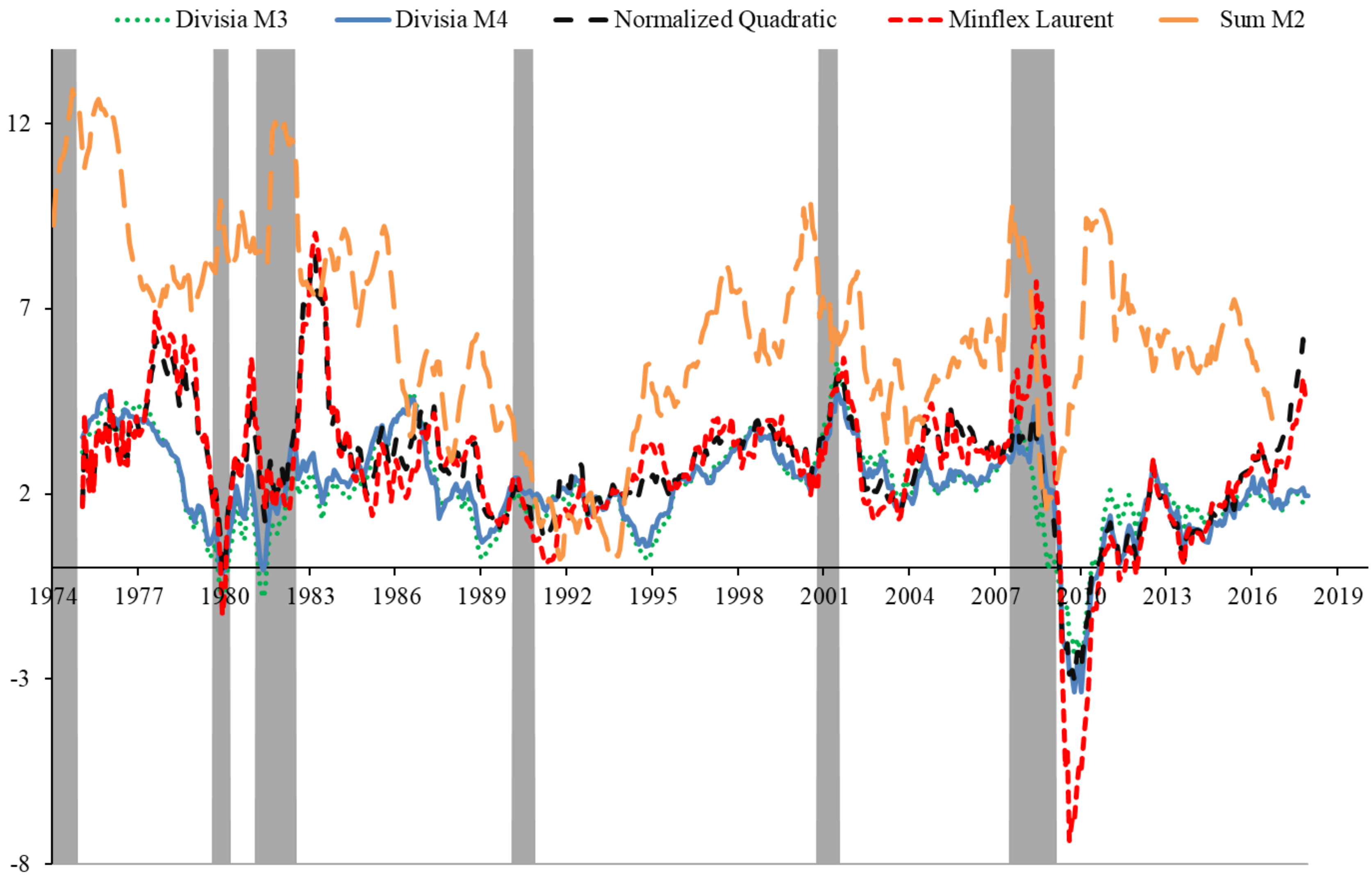


Table 3. Parameter estimates of the Ireland (2004) model

	Monetary aggregate				
	Functional monetary aggregates		Statistical monetary aggregates		
Parameter	ML	NQ	Sum M2	Divisia M3	Divisia M4
ω_2	0.0084 (0.0165)	0.0059 (0.6147)	0.0006 (0.0005)	0.0002 (0.0000)	0.0001 (0.0001)
γ_1	0.2040 (0.8802)	0.3422 (0.3409)	0.1194 (0.2237)	0.0149 (0.8904)	0.2870 (0.7246)
γ_2	0.0552 (0.1247)	0.1293 (0.2726)	0.8694 (0.2882)	1.2305 (0.3979)	1.1024 (0.3861)
γ_3	0.9993 (0.0016)	0.9984 (0.0033)	0.9894 (0.0035)	0.9850 (0.0049)	0.9865 (0.0047)
ρ_r	0.7703 (0.0222)	0.7703 (0.0224)	0.7704 (0.0219)	0.7704 (0.0195)	0.7704 (0.0219)
ρ_y	0.0154 (0.0045)	0.0154 (0.0042)	0.0153 (0.0043)	0.0153 (0.0051)	0.0153 (0.0043)
ρ_π	0.4225 (0.0349)	0.4225 (0.0294)	0.4224 (0.0332)	0.4224 (0.0302)	0.4224 (0.0323)
ρ_a	0.9558 (0.0154)	0.9558 (0.0155)	0.9558 (0.0161)	0.9958 (0.0153)	0.9558 (0.0149)
ρ_e	0.9743 (0.0160)	0.9874 (0.0116)	0.9963 (0.0040)	0.9983 (0.0024)	0.9974 (0.0043)
ρ_z	0.9903 (0.0108)	0.9903 (0.0100)	0.9904 (0.0094)	0.9904 (0.0107)	0.9904 (0.0108)
σ_a	0.0399 (0.0098)	0.0399 (0.0107)	0.0399 (0.0112)	0.0399 (0.0113)	0.0399 (0.0097)
σ_e	0.0190 (0.0010)	0.0146 (0.0008)	0.0087 (0.0004)	0.0112 (0.0006)	0.0108 (0.0006)
σ_z	0.0092 (0.0006)	0.0092 (0.0006)	0.0092 (0.0005)	0.0092 (0.0006)	0.0092 (0.0006)
σ_r	0.0028 (0.0002)	0.0028 (0.0002)	0.0028 (0.0002)	0.0028 (0.0002)	0.0028 (0.0002)

Notes: Numbers in parentheses are standard errors.

Table 4. Parameter estimates of the Andrés *et al.* (2006) model

	Monetary aggregate				
	Functional monetary aggregates		Statistical monetary aggregates		
Parameter	ML	NQ	Sum M2	Divisia M3	Divisia M4
ψ_1	0.0096 (0.0001)	0.0086 (0.0000)	0.8918 (0.0518)	0.0167 (0.0001)	0.0166 (0.0000)
ψ_2	1.3245 (0.0020)	1.3120 (0.0000)	0.0000	1.7728 (0.0048)	1.7986 (0.0047)
h	0.7109 (0.0025)	0.7299 (0.0003)	0.9682 (0.0180)	0.7929 (0.0008)	0.7904 (0.0002)
γ_1	0.8972 (0.0050)	0.9329 (0.0062)	0.0177 (0.0018)	0.7289 (0.0045)	0.7184 (0.0028)
γ_2	0.0166 (0.0000)	0.0288 (0.0000)	1.2902 (0.1551)	0.0141 (0.0000)	0.0139 (0.0000)
χ	2.4173 (0.0027)	2.6287 (0.0008)	2.6250 (0.8198)	2.6295 (0.0004)	2.6247 (0.0022)
λ	0.6805 (0.0141)	0.8132 (0.0001)	0.0520 (0.0220)	0.1011 (0.0013)	0.1020 (0.0021)
ρ_r	0.6547 (0.0105)	0.5906 (0.0001)	0.7847 (0.0250)	0.7841 (0.0138)	0.7802 (0.0103)
ρ_y	0.0244 (0.0001)	0.1182 (0.0000)	0.1215 (0.0240)	0.0397 (0.0002)	0.0373 (0.0026)
ρ_π	2.3182 (0.1057)	1.2682 (0.0003)	1.8480 (0.1935)	1.9453 (0.0101)	1.9308 (0.0009)
ρ_μ	0.1305 (0.0078)	0.1915 (0.0001)	0.5767 (0.1415)	0.1855 (0.0023)	0.1760 (0.0038)
ρ_a	0.9872 (0.0032)	0.9841 (0.0005)	0.9446 (0.0258)	0.9723 (0.0038)	0.9752 (0.0048)
ρ_e	0.9792 (0.0097)	0.9725 (0.0007)	0.9965 (0.0038)	0.9963 (0.0035)	0.9925 (0.0070)
ρ_z	0.8296 (0.0092)	0.8490 (0.0064)	0.9791 (0.0164)	0.9765 (0.0041)	0.9760 (0.0034)
σ_a	0.1946 (0.0814)	0.1364 (0.0115)	0.0274 (0.0185)	0.0540 (0.0083)	0.0593 (0.0110)
σ_e	0.0199 (0.0011)	0.0157 (0.0005)	0.0092 (0.0005)	0.0115 (0.0006)	0.0114 (0.0006)
σ_z	0.0229 (0.0017)	0.0284 (0.0014)	0.0082 (0.0011)	0.0014 (0.0001)	0.0014 (0.0002)
σ_r	0.0039 (0.0002)	0.0037 (0.0001)	0.0029 (0.0002)	0.0027 (0.0002)	0.0027 (0.0001)

Notes: Numbers in parentheses are standard errors. Unavailable standard error suggests that the non-negativity constraint binds.

Table 5. Granger causality tests

Monetary aggregate	AIC optimal lag	Statistic (p-value)
	A. 1974-2018	
ML	(12,3,4)	9.7715 (0.0001)
NQ	(7,10,4)	4.2285 (0.0000)
Sum M2	(12,1,2)	0.0002 (0.9889)
Divisia M3	(12,2,2)	8.8571 (0.0002)
Divisia M4	(12,6,5)	3.7818 (0.0011)
	B. 1974-1985	
ML	(1,12,5)	2.2857 (0.0125)
NQ	(1,12,5)	2.5520 (0.0053))
Sum M2	(1,12,5)	1.1565 (0.3239)
Divisia M3	(12,3,5)	4.2496 (0.0071)
Divisia M4	(5,1,12)	13.2275 (0.0004)
	C. 1986-1999	
ML	(3,4,2)	3.3854 (0.0109)
NQ	(3,4,2)	3.0913 (0.0175)
Sum M2	(3,1,2)	0.2493 (0.6182))
Divisia M3	(3,1,2)	1.1315 (0.2890)
Divisia M4	(3,3,2)	1.5731 (0.1980)
	D. 2000-2018	
ML	(7,1,10)	0.6894 (0.4073)
NQ	(4,5,1)	3.8484 (0.0023)
Sum M2	(7,1,10)	0.3566 (0.5511)
Divisia M3	(7,1,10)	0.0162 (0.8990)
Divisia M4	(7,1,10)	0.0098 (0.9211)

Note: Numbers in parentheses are p-values.

Figure 5. Probabilities of the ML functional monetary aggregate causing real output

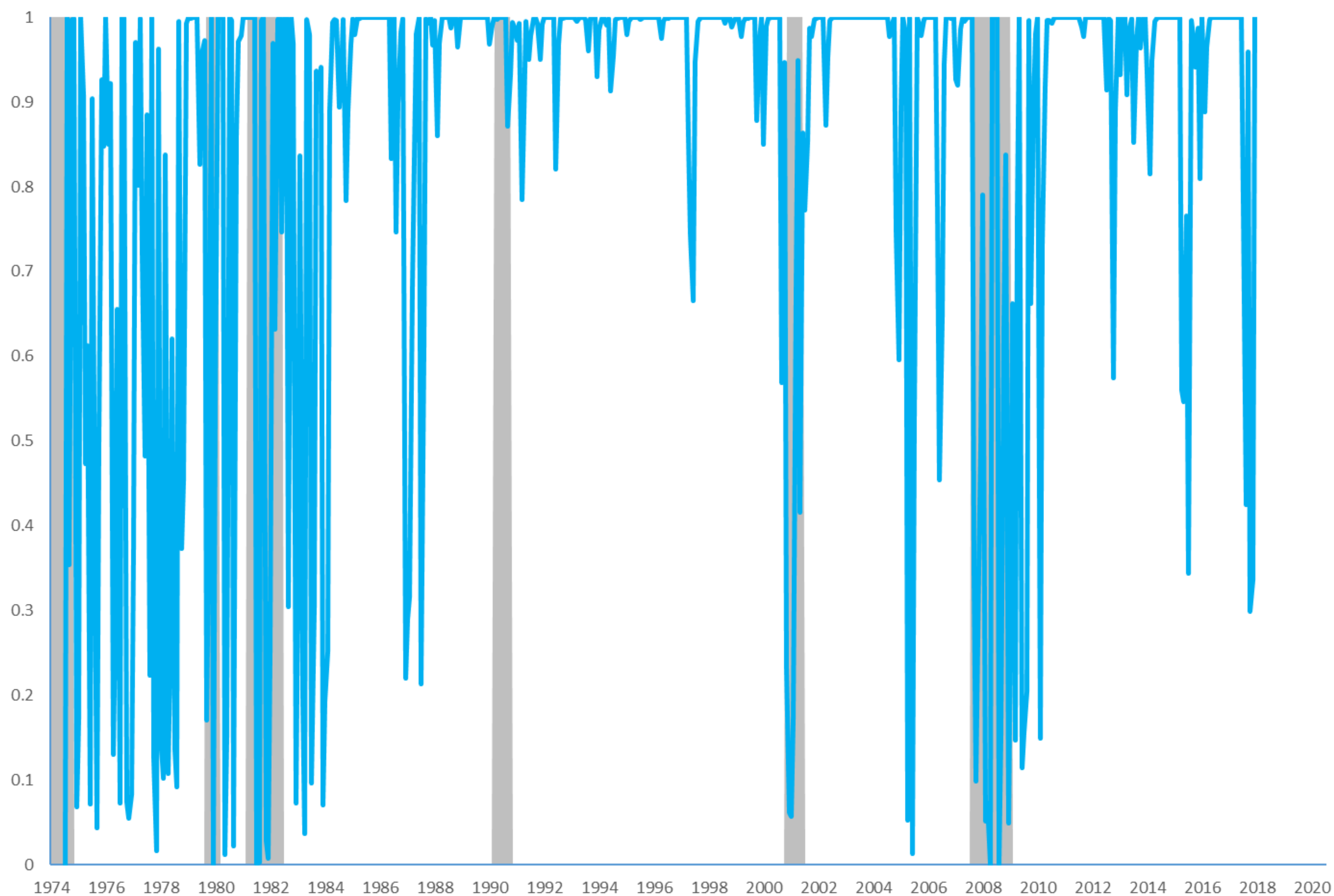


Figure 6. Probabilities of the NQ functional monetary aggregate causing real output

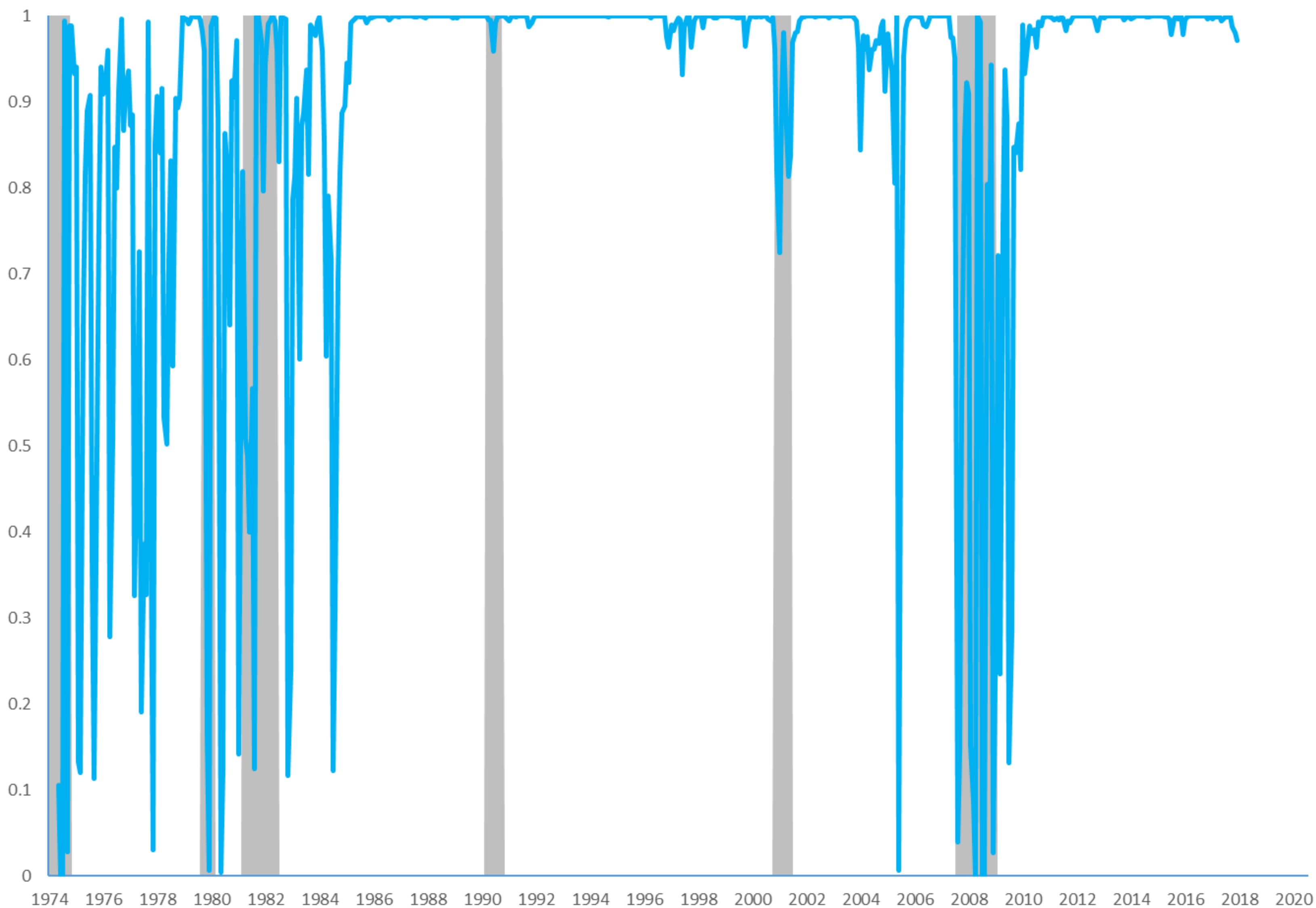
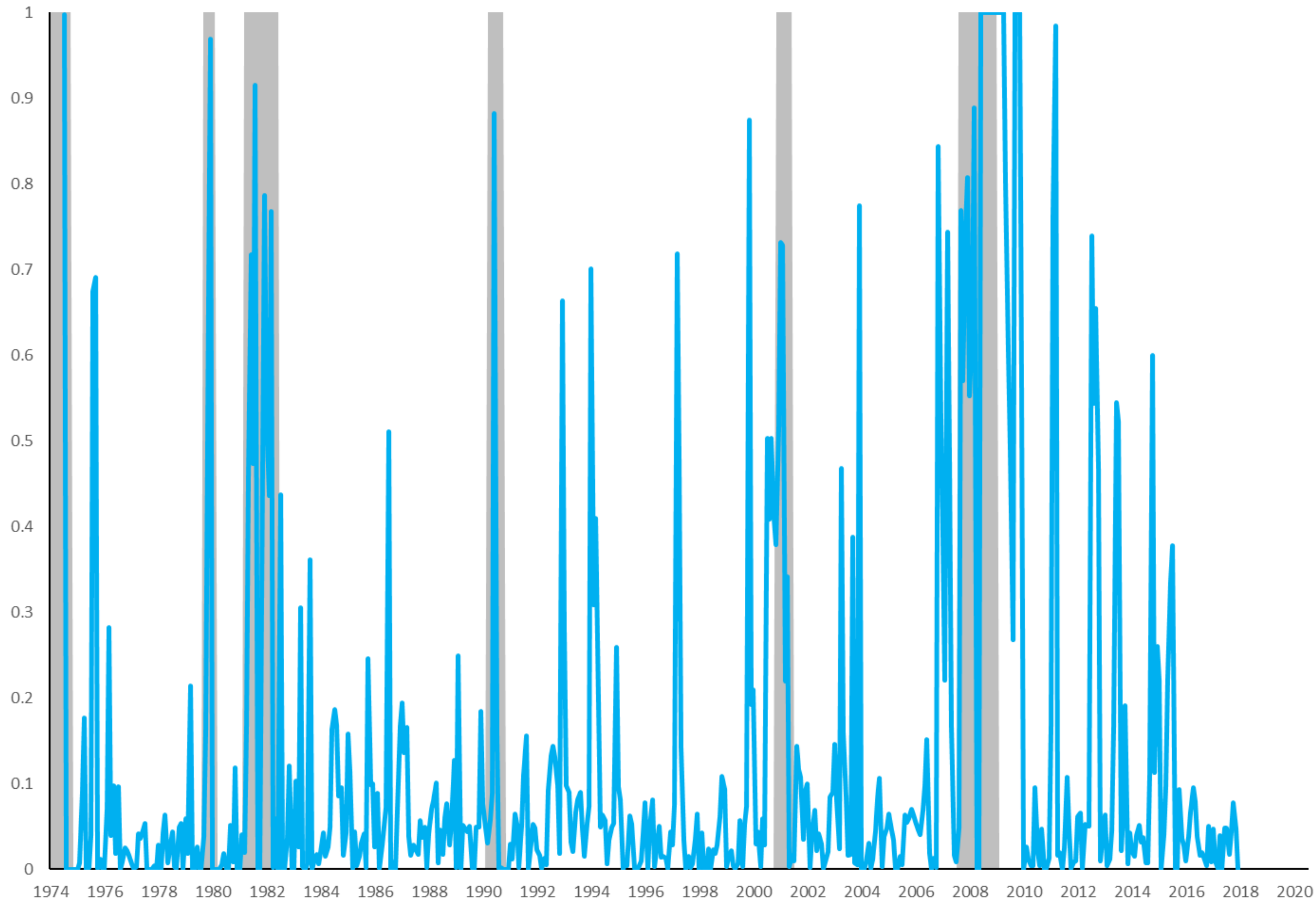


Figure 7. Probabilities of the Sum M2 monetary aggregate causing real output



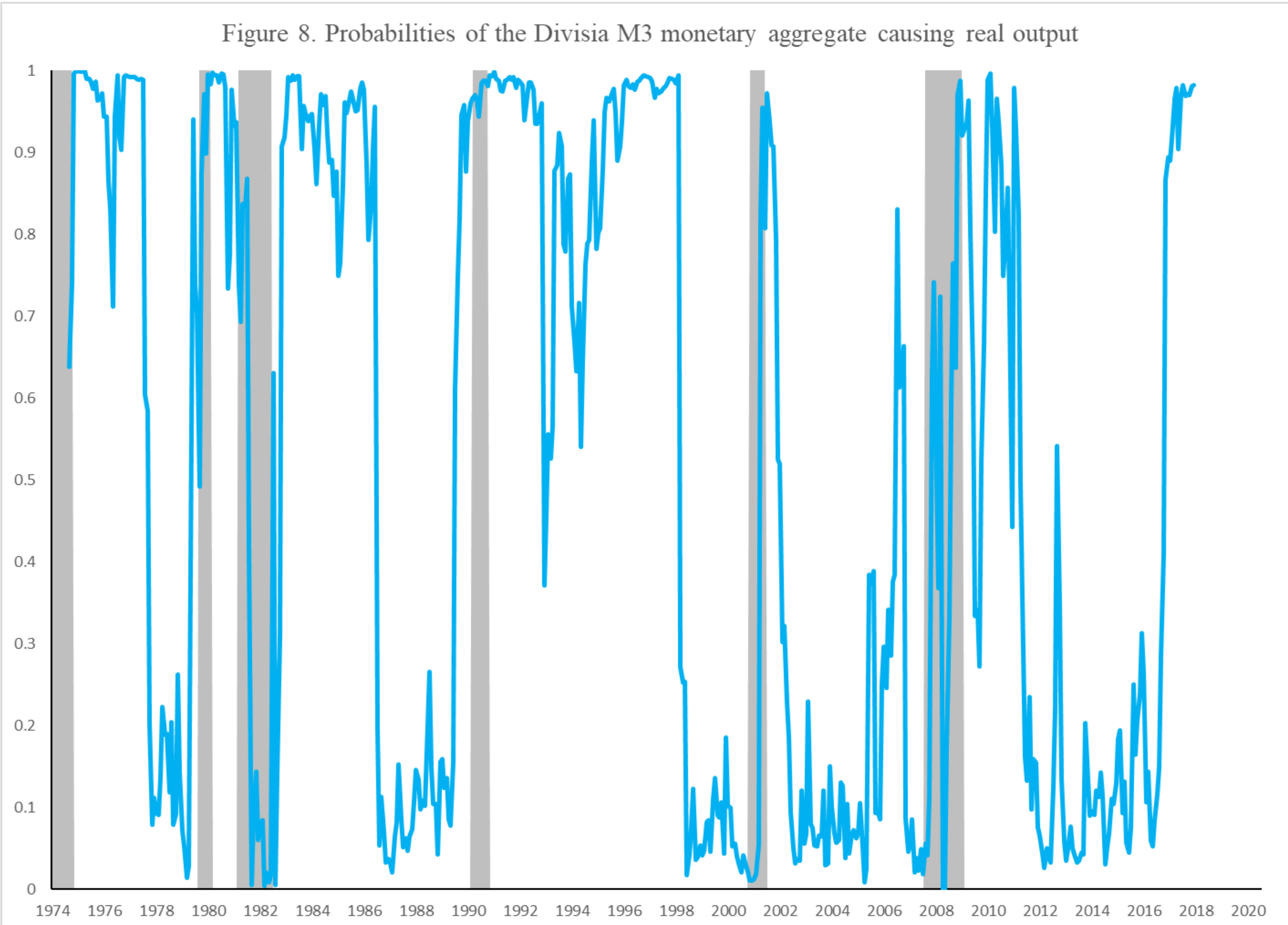


Figure 8. Probabilities of the Divisia M3 monetary aggregate causing real output
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1974
1976
1978
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
2012
2014
2016
2018
2020

Figure 9. Probabilities of the Divisia M4 monetary aggregate causing real output