Identification and Estimation of Dynamic Games when Players' Beliefs Are Not in Equilibrium

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Abstract

This paper deals with the identification and estimation of dynamic games when players' beliefs about other players' actions are biased, i.e., beliefs do not represent the probability distribution of the actual behavior of other players conditional on the information available. First, we show that a exclusion restriction, typically used to identify empirical games, provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs. Second, we prove that this exclusion restriction, together with consistent estimates of beliefs at several points in the support of the *special* state variable (i.e., the variable involved in the exclusion restriction), is sufficient for nonparametric point-identification of players' payoff and belief functions. The consistent estimates of beliefs at some points of support may come either from an assumption of unbiased beliefs at these points in the state space, or from available data on elicited beliefs for some values of the state variables. Third, we propose a simple two-step estimation method and a sequential generalization of the method that improves its asymptotic and finite sample properties. We illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains. The key conditions for the identification of beliefs and payoffs in our application are the following: (a) the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit because the existence of sunk entry costs and economies of density in these costs; and (b) firms' beliefs are unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores, though beliefs are unrestricted, and potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs.

Keywords: Dynamic games; Rational behavior; Rationalizability; Identification; Estimation; Market entry-exit.

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1 Introduction

The principle of *revealed preference* (Samuelson, 1938) is a cornerstone in the empirical analysis of decision models, either static or dynamic, single-agent problems or games. Under the principle of *revealed preference*, agents maximize expected payoffs and their actions reveal information on the structure of payoff functions. This simple but powerful concept has allowed econometricians to use data on agents' decisions to identify important structural parameters for which there is very limited information from other sources. Examples of parameters and functions that have been estimated using the principle of revealed preference are agents' degree of risk aversion, intertemporal rates of substitution, market entry costs, adjustment costs and switching costs, consumer willingness to pay, preference for a political party, or the benefits of a merger. In the context of empirical games, where players' expected payoffs depend on their beliefs about the behavior of other players, most applications combine the principle of revealed preference with the assumption that players' beliefs about the behavior of other players are *in equilibrium*, in the sense that these beliefs represent the probability distribution of the actual behavior of other players conditional on the information available. Equilibrium beliefs plays an important role in the identification and estimation of games, and as such are almost always assumed in the empirical game literature. Equilibrium restrictions have identification power even in models with multiple equilibria (Tamer, 2003, Aradillas-Lopez and Tamer, 2008, Bajari, Hong, and Ryan, 2010). Imposing these restrictions contributes to improved asymptotic and finite sample properties of game estimators. Moreover, the assumption of equilibrium beliefs are very useful for evaluating counterfactual policies in a strategic environment. Models where agents' beliefs are endogenously determined in equilibrium not only take into account the direct effect of the new policy on agents' behavior through their payoff functions, but also through an endogenous change in agents' beliefs.

Despite the clear benefit that the assumption of equilibrium beliefs delivers to an applied researcher, we can think of at least three important examples where the assumption is not realistic and it is of interest to relax it. First, competition in oligopoly industries is often characterized by strategic uncertainty (Besanko et al., 2010). Firm managers are very secretive about their own strategies and face significant uncertainty about the strategies of their competitors. In fact, it is often the case that firms have incentives to misrepresent their own strategies.[1] In this context, it may be unrealistic to expect firms to have unbiased beliefs about competitors' behavior. Second, although the assumption of equilibrium beliefs is potentially useful for policy evaluation in a strategic environment, it can also be unrealistic. Suppose that to evaluate a policy change we estimate an empirical game using data before and after the policy is implemented. If the policy change is substantial, it would seem reasonable to allow for some period of strategic uncertainty immediately following the policy change. Players (e.g., firms) will be uncertain about their competitors' strategies and it will take time to adjust to the new equilibrium. Thus, at least for some period of time, firms' beliefs will be out of equilibrium, and imposing the restriction of equilibrium beliefs may bias

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[1]For example, it is in the interest of a firm for its competitors to believe that it is planning an expansion in a particular location to deter entry when in fact there is no such plan. See also Morris and Song (2002) for examples of models with strategic uncertainty and related experimental evidence.

the estimates of the effects of the new policy. A third example comes from the structural estimation of games using data generated by laboratory experiments. It is well established in the experimental economics literature on games that there is significant heterogeneity in players' elicited beliefs, and that this heterogeneity is often one of the most important factors in explaining heterogeneity in observed behavior in the laboratory.[2] Imposing the assumption of equilibrium beliefs in these applications does not seem reasonable. Interestingly, however, recent empirical papers establish a significant divergence between stated or elicited beliefs and the beliefs inferred from players' actions using, for example, revealed preference-based methods (see Costa-Gomes and Weizsäcker, 2008, and Rutström and Wilcox, 2009). The results in our paper can be applied to estimate beliefs and payoffs, using either observational or laboratory data, when the researcher wants to allow for the possibility of biased beliefs but he does not have data on elicited beliefs, or data on elicited beliefs is limited to only a few states of the world.

In this paper we study nonparametric identification, estimation, and inference in dynamic discrete games of incomplete information when we assume that players are rational, in the sense that each player takes an action that maximizes his expected payoff given some beliefs, but we relax the assumption that these beliefs are in equilibrium. In the class of models that we consider, a player's belief is a probability distribution over the space of other players' actions conditional on some state variables, or the player's information set. Beliefs are biased, or not in equilibrium, if they are different from the actual probability distribution of other players' actions conditional on the state variables of the model. We consider a nonparametric specification of beliefs and treat these probability distributions as *incidental parameters* that, together with the structural parameters in payoff functions and transition probabilities, determine the stochastic process followed by players' actions.[3]

Note that, since players condition their beliefs at any point in time on the information available to them, to define beliefs as biased or unbiased a researcher must first specify what information is available to players. Therefore, a potential reason why players' beliefs may appear biased is that players' beliefs are conditioned on an information set different from the one postulated by the researcher. In this sense, one attractive feature of the framework that we propose in this paper is that it can be interpreted as an approach to the estimation of games of incomplete information that is robust to misspecification of the information available to players when they form their beliefs about other players.

When players beliefs are not in equilibrium they are by definition different from the actual distribution of players' actions in the population. Therefore, without other restrictions, beliefs cannot be identified, or consistently estimated, by simply using a nonparametric estimator of the distribution of players' actions conditional on the state variables. First, we show that an exclusion restriction that is typically used to identify payoffs in empirical games (Bajari, Hong, and Ryan, 2010, Bajari et al., 2011), provides testable nonparametric restrictions of the null hypothesis of

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[2]See Camerer (2003) and recent papers by Costa-Gomes and Weizsäcker (2008), and Palfrey and Wang (2009).

[3]Our framework includes as particular case games with multiple equilibria where every player has beliefs that correspond to an equilibrium but their beliefs are not 'synchronized', i.e., some players believe that the game is in an equilibrium, say A, and other players think that the game is in a different equilibrium, say B.

equilibrium beliefs. This type of exclusion restriction assumes there is a state variable which enters the payoffs of one player directly, but is *excluded* from the payoffs of other players; it may only enter the payoffs of other players indirectly through their beliefs. Second, we prove that this exclusion restriction, together with consistent estimates of beliefs at two points in the support of the *special* state variable (i.e., the variable that satisfies the exclusion restriction), is sufficient for nonparametric point-identification of players' payoff and belief functions. The consistent estimates of beliefs at two points of support may come either from an assumption of unbiased beliefs at these points in the state space, or from data on elicited beliefs for some values of the state variables. Third, we propose a simple two-step estimation method and a sequential generalization of the method that improves its asymptotic and finite sample properties. The two-step method has an analogy to instrumental variables estimation in regression models, and we use this analogy to discuss the identification power of our restrictions and the potential problem of weak instruments in applications. Finally, we illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains.

Monte Carlo experiments help to understand the trade-off a researcher faces when deciding whether or not to impose the assumption of equilibrium beliefs in an application, and how this trade off depends on properties of the underlying data generating process. On the one hand, the assumption of equilibrium beliefs affords the researcher significant identification power, which translates into precise estimates of payoffs and beliefs. On the other hand, imposing the assumption of equilibrium beliefs in estimation when beliefs in the underlying data generating process are in fact not in equilibrium may result in biased estimates of payoffs and beliefs. We find that while there is a loss in precision when we drop the assumption of equilibrium beliefs, this loss is larger when beliefs in the underlying DGP are actually in equilibrium. That is, the cost of dropping the assumption of equilibrium beliefs is larger if beliefs truly are in equilibrium than if they are not. We also find that the bias associated with incorrectly assuming equilibrium beliefs is very significant. Estimates of beliefs are biased by up to 100% of their true value, and estimates of payoff parameters are in some cases biased by over 60% of their true value. We also provide some evidence on how identification of payoffs and beliefs is affected by how useful the "special" excluded variable is as an instrument. This variable acts to shift one player's payoffs exogenously, while only affecting the other player through his beliefs. We provide evidence that as the quality of this instrument improves, the mean-squared error of estimates can be significantly reduced.

This paper builds on the recent literature on estimation of dynamic games of incomplete information (see Aguirregabiria and Mira, 2007, Bajari, Benkard and Levin, 2007, Pakes, Ostrovsky and Berry, 2007, and Pesendorfer and Schmidt-Dengler, 2008). All the papers in this literature assume that the data come from a Markov Perfect Equilibrium. We relax that assumption. Our research also builds upon and extends the work of Aradillas-Lopez and Tamer (2008) who study the identification power of the assumption of equilibrium beliefs in simple static games. We extend their work in several ways. First, we study dynamic games, including static games as a particular case. The implications of dropping the assumption of equilibrium beliefs in dynamic games, with respect to identification in particular, are quite different from those of static games. As we show

in this paper, the characterization and derivation of bounds on choice probabilities is significantly more complicated in dynamic games, and the key identification results in Aradillas-Lopez and Tamer cannot be directly extended to the case of dynamic games. Therefore, we follow a different approach from the one considered by Aradillas-Lopez and Tamer. Second, while their study is focused primarily on identification, we propose and implement new tests and estimators and study their properties. And third, they consider identification of parametrically specified models, while our point of departure is nonparametric identification of payoffs and beliefs.

Our approach also differs from Aradillas-Tamer in one key aspect. In relaxing the assumption of Nash equilibrium, they consider a very specific departure from equilibrium beliefs. They assume that players are *level-k rational* with respect to their beliefs about their opponents' behavior, a concept which derives from the notion of rationalizability (Bernheim, 1984, and Pearce, 1984). Their approach is especially useful in the context of static games with binary or ordered decision variables, as, under the condition that players' payoffs are monotone in the decision of their opponents, it yields a sequence of closed form bounds on players' beliefs that grow tighter as the level of rationality k gets larger. Unfortunately, in the case of dynamic games, the assumptions of Aradillas-Lopez and Tamer do not yield a representation of bounds on players' beliefs that is practical to implement, even for simple dynamic games. We describe this issue at the end of section 2. As such we do not use a bound-approach that relies on the notion of level-k rationalizability. Instead, we concentrate on level-1 rationalizability and study conditions and methods for nonparametric point identification and estimation of preferences and beliefs.

Our paper also complements the growing literature on the use of data on subjective expectations in microeconometric decision models, especially the contributions of Walker (2003), Manski (2004), Delavande (2008), and Van der Klaauw and Wolpin (2008). It is commonly the case that data on elicited beliefs has the form of unconditional probabilities, or probabilities that are conditional only on a strict subset of the state variables in the postulated model. In this context, the framework that we propose in this paper can be combined with the incomplete data on elicited beliefs in order to obtain nonparametric estimates of the complete conditional probability distribution describing an individual's beliefs.

To illustrate our model and methods in the context of an empirical application, we consider a dynamic game of store location between McDonalds and Burger King. There has been very little work on the bounded rationality of firms, as most empirical studies on bounded rationality have concentrated on individual behavior.[4] They key conditions for the identification of beliefs and payoffs in our application are the following. The first condition is an exclusion restriction in a firm's profit function that establishes that the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit because the existence of sunk entry costs and economies of density in these costs. The second condition restricts firms' beliefs to be unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores. However, beliefs are unrestricted, and

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[4]An exception is the recent paper by Goldfarb and Xiao (2011) that studies entry decisions in the US local telephone industry and finds significant heterogeneity in firms' beliefs about other firms' strategic behavior.

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potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs for Burger King. More specifically, we find that this firm underestimated the probability of entry of McDonalds in markets that were relatively far away from McDonalds' network of stores.

The rest of the paper includes the following sections. Section 2 presents the model and basic assumptions. In section 3, we present our identification results. Section 4 describes estimation methods and testing procedures. Section 5 presents our Monte Carlo experiments. The empirical application is described in section 6. We summarize and conclude in section 7.

2   Model

2.1   Basic framework

This section presents a dynamic game of incomplete information where N players make discrete choices over T periods. We use indexes $i, j \in \{1, 2, ...N\}$ to represent players, and the index $-i$ to represent all players other than i. Time is discrete and indexed by $t \in \{1, 2, ..., T\}$. The time horizon T can be either finite or infinite. Every period t, players choose simultaneously one out of A alternatives from the choice set $\mathcal{Y} = \{0, 1, ..., A - 1\}$. Let $Y_{it} \in \mathcal{Y}$ represent the choice of player i at period t. Each player makes this decision to maximize his expected and discounted payoff, $E_t(\sum_{s=0}^{T} \beta^s \Pi_{i,t+s})$, where $\beta \in (0, 1)$ is the discount factor, and Π_{it} is his payoff at period t. The one-period payoff function has the following structure:

$$\Pi_{it} = \pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it}) \tag{1}$$

$\mathbf{Y}_{-it}$ represents the current action of the other players; $\mathbf{X}_t$ is a vector of state variables which are common knowledge for both players; $\boldsymbol{\varepsilon}_{it} \equiv (\varepsilon_{it}(0), \varepsilon_{it}(1), ..., \varepsilon_{it}(A))$ is a vector of private information variables for firm i at period t; and $\pi_{it}(.)$ is a real valued function.

The vector of common knowledge state variables is $\mathbf{X}_t$, and it evolves over time according to the transition probability function $f_t(\mathbf{X}_{t+1} | \mathbf{Y}_t, \mathbf{X}_t)$ where $\mathbf{Y}_t \equiv (Y_{1t}, Y_{2t}, ..., Y_{Nt})$. The vector of private information shocks $\boldsymbol{\varepsilon}_{it}$ is independent of $\mathbf{X}_t$ and independently distributed over time and players. Without loss of generality, these private information shocks have zero mean. The distribution function of $\boldsymbol{\varepsilon}_{it}$ is given by G_{it}, which is absolutely continuous and strictly increasing with respect to the Lebesgue measure on $\mathbb{R}^A$. When the game has infinite horizon ($T = \infty$), we assume that all the primitive functions, π_{it}, G_{it}, and f_t, are constant over time such that the dynamic game has a stationary Markov structure.

EXAMPLE 1: Dynamic game of market entry and exit. Consider N firms competing in a market. Each firm sells a differentiated product. Every period, firms decide whether or not to be active in the market. Then, incumbent firms compete in prices. Let $Y_{it} \in \{0, 1\}$ represent the decision of firm i to be active in the market at period t. The profit of firm i at period t has the structure of equation (1), $\Pi_{it} = \pi_{it}(Y_{it}, Y_{-it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it})$. We now describe the specific form of the payoff function π_{it} and the state variables $\mathbf{X}_t$ and ε_{it}. The average profit of an inactive firm, $\pi_{it}(0, Y_{jt}, \mathbf{X}_t)$,

is normalized to zero, such that $\Pi_{it} = \varepsilon_{it}(0)$. The profit of an active firm is $\pi_{it}(1, Y_{jt}, \mathbf{X}_t) + \varepsilon_{it}(1)$ where:

$$
\begin{aligned}
\pi_{it}(1, Y_{jt}, \mathbf{X}_t) \quad = \quad & H_t \left(\theta_i^M - \theta_i^D \sum_{j \neq i} Y_{jt} \right) \\
& - \theta_{i0}^{FC} - \theta_{i1}^{FC} \ Z_i - 1\{Y_{it-1} = 0\} \ \theta_i^{EC}
\end{aligned}
\tag{2}
$$

The term $H_t \left(\theta_i^M - \theta_i^D \sum_{j \neq i} Y_{jt} \right)$ is the variable profit of firm i. H_t represents market size (e.g., market population) and it is an exogenous state variable. θ_i^M is a parameter that represents the per capita variable profit of firm i when the firm is a monopolist. The parameter θ_i^D captures the effect of the number of competing firms on the profit of firm i.[5] The term $\theta_{i0}^{FC} + \theta_{i1}^{FC} \ Z_i$ is the fixed cost of firm i, where θ_{i0}^{FC} and θ_{i1}^{FC} are parameters, and Z_i is an exogenous, time-invariant, firm characteristic affecting the fixed cost of the firm. The term $1\{Y_{it-1} = 0\} \ \theta_i^{EC}$ represents sunk entry costs, where $1\{.\}$ is the binary indicator function and θ_i^{EC} is a parameter. Entry costs are paid only if the firm was not active in the market at previous period. The vector of common knowledge state variables of the game is $\mathbf{X}_t = (H_t, Z_i, Y_{it-1} : i = 1, 2, ..., N)$. ∎

Most previous literature on estimation of dynamic discrete games assumes that the data comes from a Markov Perfect Equilibrium (MPE). This equilibrium concept incorporates four main assumptions.

ASSUMPTION MOD-1 (Payoff relevant state variables): Players' strategy functions depend only on payoff relevant state variables: $\mathbf{X}_t$ and $\boldsymbol{\varepsilon}_{it}$. Also, a player's belief about the strategy of other player is a function only of the payoff relevant state variables of the other player.

ASSUMPTION MOD-2 (Maximization of expected payoffs): Players are forward looking and maximize expected intertemporal payoffs.

ASSUMPTION MOD-3 (Unbiased beliefs on own future behavior): A player's beliefs about his own actions in the future are unbiased expectations of his actual actions in the future.

ASSUMPTION 'EQUIL' (Unbiased or equilibrium beliefs on other players' behavior): Strategy functions are common knowledge, and players' have rational expectations on the current and future behavior of other players. That is, players' beliefs about other players' actions are unbiased expectations of the actual actions of other players.

First, let us examine the implications of imposing only Assumption MOD-1. The payoff-relevant information set of player i is $\{\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}\}$. The space of $\mathbf{X}_t$ is $\mathcal{X}$. At period t, players observe $\mathbf{X}_t$ and choose their respective actions. Let the function $\sigma_{it}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}) : \mathcal{X} \times \mathbb{R}^A \to \mathcal{Y}$ represent a strategy function for player i at period t. Given any strategy function σ_{it}, we can define a choice probability function $P_{it}(y|\mathbf{X}_t)$ that represents the probability that $Y_{it} = y$ conditional on $\mathbf{X}_t$ given that player i follows strategy σ_{it}. That is,

$$
P_{it}(y|\mathbf{X}_t) \equiv \int 1 \left\{ \sigma_{it} \left(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it} \right) = y \right\} \ dG_{it} \left(\boldsymbol{\varepsilon}_{it} \right)
\tag{3}
$$

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[5] A more flexible specification allows for each firm j to have a different impact on the variable profit of firm i, i.e., $W_t \left(\theta_i^M - \sum_{j \neq i} \theta_{ij}^D \ Y_{jt} \right)$.

It is convenient to represent players' behavior using these *Conditional Choice Probability* (CCP) functions. When the variables in $\mathbf{X}_t$ have a discrete support, we can represent the CCP function $P_{it}(.)$ using a finite-dimensional vector $\mathbf{P}_{it} \equiv \{P_{it}(y|\mathbf{X}_t) : y \in \mathcal{Y}, \mathbf{X}_t \in \mathcal{X}\}$. Throughout the paper we use either the function $P_{it}(.)$ or the vector $\mathbf{P}_{it}$ to represent the *actual behavior* of player i at period t.

Without imposing Assumption 'Equil' ('Equilibrium Beliefs'), a player's beliefs about the behavior of other players do not necessarily represent the actual behavior of the other players. Therefore, we need functions other than $\sigma_{jt}(.)$ and $P_{jt}(.)$ to represent players i's beliefs about the strategy of other players. Let $b_{it}^{(t_0)}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{-it})$ be a function from $\mathcal{X} \times \mathbb{R}^{(N-1)A}$ into $\mathcal{Y}^{N-1}$ that represents player i's belief at period t_0 about the strategy function of all the other players at period t. In principle, this function may vary with t_0 due to players' learning and forgetting, or to other factors that cause players' beliefs to change over time. Let $B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X}_t)$ be the choice probability associated with $b_{it}^{(t_0)}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{-it})$, i.e., $B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X}_t) \equiv \int 1\{b_{it}^{(t_0)}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{-it}) = \mathbf{y}_{-i}\} \, dG_{-it}(\boldsymbol{\varepsilon}_{-it})$. When $\mathcal{X}$ is a discrete and finite space, we can represent function $B_{it}^{(t_0)}(.)$ using a finite-dimensional vector $\mathbf{B}_{it}^{(t_0)} \equiv \{B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X})) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}, \mathbf{X} \in \mathcal{X}\}$. Using this notation, Assumption 'Equil' can be represented in vector form as $\mathbf{B}_{it}^{(t_0)} = \Pi_{j \neq i} \mathbf{P}_{jt}$ for every player i, every t_0, and $t \geq t_0$.

The following assumption replaces the assumption of 'Equilibrium Beliefs' and summarizes our minimum conditions on players' beliefs.

ASSUMPTION MOD-4: If the dynamic game has finite horizon ($T < \infty$), then players' beliefs functions $B_{it}^{(t_0)}$ may vary over the time period of the opponent's behavior, t, but they are not revised or updated over t_0, i.e., $B_{it}^{(t_0)} = B_{it}$ for any period $t_0 \leq t$. If the dynamic game has infinite horizon ($T = \infty$), then players' beliefs functions $B_{it}^{(t_0)}$ may be revised over t_0, but they do not vary over time t because the decision problem is stationary, i.e., $B_{it}^{(t_0)} = B_i^{(t_0)}$ for every period t.

Assumption MOD-4 imposes restrictions on the time pattern of beliefs. Using Table 1, we can describe this assumption by saying that beliefs are constant either across columns or across rows. For finite horizon dynamic games, we assume that beliefs are constant across rows. This implies that each player believes his opponents' behavior may change over time because the decision problem is non-stationary (finite horizon), but beliefs about opponents' behavior at a given period are constant over the entire game and they are not revised as time goes by. Therefore, for finite horizon games we do not allow for updating of beliefs. For infinite horizon games, we assume that players' know that the game is stationary and their beliefs satisfy this stationarity condition. However, players can revise their beliefs over time.

ASSUMPTION MOD-5: The state space $\mathcal{X}$ is discrete and finite, and $|\mathcal{X}|$ represents its dimension or number of elements.

For the rest of the paper, we maintain Assumptions MOD-1 to MOD-5 but we do not impose the restriction of 'Equilibrium Beliefs'. We assume that players are 'rational', in the sense that they maximize expected and discounted payoff given their beliefs on other players' behavior. Our approach is agnostic about the formation of players' beliefs.

For the sake of simplicity in the presentation of our results, the main text of the paper deals with finite horizon games, but we show in the Appendix that our results apply to infinite horizon dynamic games. To illustrate both cases, we consider a finite horizon game in the Monte Carlo experiments in section 5, and an infinite horizon game in the empirical application in section 6.

2.2 Best response mappings

We say that a strategy function σ_{it} (and the associated CCP function P_{it}) is *rational* if for every possible value of $(\mathbf{X}_t, \varepsilon_{it}) \in \mathcal{X} \times \mathbb{R}^A$ the action $\sigma_{it}(\mathbf{X}_t, \varepsilon_{it})$ maximizes player i's expected and discounted value given his beliefs on the opponent's strategy. Given his beliefs, player i's best response at period t is the optimal solution of a single-agent dynamic programming (DP) problem. This DP problem can be described in terms of: (i) a discount factor, β; (ii) a sequence of expected one-period payoff functions, $\{\pi_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it}) : t = 1, 2, ..., T\}$, where

$$\pi_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) \equiv \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} \pi_{it}(Y_{it}, \mathbf{y}_{-i}, \mathbf{X}_t) \; B_{it}(\mathbf{y}_{-i}|\mathbf{X}_t) \; ; \tag{4}$$

and (iii) a sequence of transition probability functions $\{f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) : t = 1, 2, ..., T\}$, where

$$f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) \;\; = \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} f_t(\mathbf{X}_{t+1}|Y_{it}, y_{-i}, \mathbf{X}_t) \; B_{it}(\mathbf{y}_{-i}|\mathbf{X}_t) \tag{5}$$

Let $V_{it}^{\mathbf{B}}(\mathbf{X}_t, \varepsilon_{it})$ be the value function for player i's DP problem given his beliefs. By Bellman's principle, the sequence of value functions $\{V_{it}^{\mathbf{B}} : t = 1, 2, ..., T\}$ can be obtained recursively using backwards induction in the following *Bellman equation*:

$$V_{it}^{\mathbf{B}}(\mathbf{X}_t, \varepsilon_{it}) \;\; = \;\; \max_{Y_{it} \in \mathcal{Y}} \left\{ \; v_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it}) \; \right\} \tag{6}$$

where $v_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t)$ is the *conditional choice value function*

$$v_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) \;\; \equiv \;\; \pi_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) + \beta \int V_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1}, \varepsilon_{it+1}) \; dG_{it}(\varepsilon_{it+1}) \; f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) \tag{7}$$

Given his beliefs, the best response function of player i at period t is the optimal decision rule of this DP problem. This best response function can be represented using the following threshold condition:

$$\{Y_{it} = y\} \text{ iff } \left\{ \varepsilon_{it}(y') - \varepsilon_{it}(y) \leq v_{it}^{\mathbf{B}}(y, \mathbf{X}_t) - v_{it}^{\mathbf{B}}(y', \mathbf{X}_t) \;\; \text{for any } y' \neq y \right\} \tag{8}$$

The *best response probability function* (*BRPF*) is a probabilistic representation of the best response function. More precisely, it is the best response function integrated over the distribution of ε_{it}. In this model, the *BRPF* is:

$$\Pr(Y_{it} = y|\mathbf{X}_t) \;\; = \;\; \int \mathbf{1} \left\{ \varepsilon_{it}(y') - \varepsilon_{it}(y) \leq v_{it}^{\mathbf{B}}(y, \mathbf{X}_t) - v_{it}^{\mathbf{B}}(y', \mathbf{X}_t) \text{ for any } y' \neq y \right\} \; dG_{it}(\varepsilon_{it})$$

$$= \;\; \Lambda_{it} \left(y; \; \widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X}_t) \right)$$

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where $\Lambda_{it}(y; .)$ is the CDF of the vector $\{\varepsilon_{it}(y') - \varepsilon_{it}(y) : y' \neq y\}$ and $\widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X}_t)$ is the $(A-1) \times 1$ vector of value differences $\{\widetilde{v}_{it}^{\mathbf{B}}(y, \mathbf{X}_t) : y = 1, 2, ..., A-1\}$ with $\widetilde{v}_{it}^{\mathbf{B}}(y, \mathbf{X}_t) \equiv v_{it}^{\mathbf{B}}(y, \mathbf{X}_t) - v_{it}^{\mathbf{B}}(0, \mathbf{X}_t)$. For instance, if $\varepsilon_{it}(y)$'s are iid Extreme Value type 1, the best response function has the well–known logit form:

$$\frac{\exp\left\{\widetilde{v}_{it}^{\mathbf{B}}(y, \mathbf{X}_t)\right\}}{\sum_{y' \in \mathcal{Y}} \exp\left\{\widetilde{v}_{it}^{\mathbf{B}}(y', \mathbf{X}_t)\right\}} \tag{9}$$

Therefore, under Assumptions MOD-1 to MOD-3 the actual behavior of player i, represented by the CCP function $P_{it}(.)$, satisfies the following condition:

$$P_{it}(y|\mathbf{X}_t) = \Lambda_{it}\left(y; \; \widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X}_t)\right) \tag{10}$$

This equation summarizes all the restrictions that Assumptions MOD-1 to MOD-3 impose on players' choice probabilities. The right hand side of equation (10) is the best response function of a rational player. We use $\Psi_{it}(\mathbf{B}_i)$ to represent in a vector form the mapping $\Lambda_{it}\left(y; \; \widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X}_t)\right)$ for every value $(y; \mathbf{X}_t)$.

The concept of Markov Perfect Equilibrium (MPE) is completed with assumption 'Equil' ('Equilibrium Beliefs'). Under this assumption, players' beliefs are in equilibrium, i.e., $\mathbf{B}_{ijt} = \mathbf{P}_{jt}$ for every pair of players i, j and every period t. A MPE can be described as a sequence of CCP vectors, $\{\mathbf{P}_{it} : i = 1, 2, ..., N; \; t = 1, 2, ..., T\}$ such that for every player i and time period t, we have that

$$\mathbf{P}_{it} = \Psi_{it}(\mathbf{P}_{-i}) \tag{11}$$

2.3 Aradillas-Lopez and Tamer's approach in dynamic games

The purpose of this subsection is twofold. First, we want to describe the relationship between our framework and the one in Aradillas-Lopez and Tamer (2008). Second, we explain in some detail why their approach, while useful for identification and estimation of static binary choice games, has very limited applicability to dynamic games.

Aradillas-Lopez and Tamer consider a static, two-player, binary-choice game of incomplete information. The model they consider can be seen as a specific case of our framework. To see this, consider the final period of the game T in our model. For the sake of notational simplicity, we omit here the vector of state variables $\mathbf{X}$ as an argument of payoff and belief functions. At the last period T, the decision problem facing the players is equivalent to that of a static game. At period T there is no future and the difference between the conditional choice value functions is simply the difference between the conditional choice current profits. For the binary choice game, there is only one difference between current profits: $\pi_{iT}^{\mathbf{B}}(1) - \pi_{iT}^{\mathbf{B}}(0)$. And taking into account that the game has only two players, we have $\pi_{iT}^{\mathbf{B}}(1) - \pi_{iT}^{\mathbf{B}}(0)$ is equal to $B_{iT}(0)\left[\pi_{iT}(1,0) - \pi_{iT}(0,0)\right] + B_{iT}(1)\left[\pi_{iT}(1,1) - \pi_{iT}(0,1)\right]$. Therefore, the *BRPF* is:

$$P_{iT}(1) = \Lambda_{iT}\left(\; B_{iT}(0)\left[\pi_{iT}(1,0) - \pi_{iT}(0,0)\right] + B_{iT}(1)\left[\pi_{iT}(1,1) - \pi_{iT}(0,1)\right]\;\right) \tag{12}$$

Aradillas-Lopez and Tamer assume that players' payoffs are submodular in players' decisions (Y_i, Y_j), i.e., for every value of the state variables $\mathbf{X}$,

$$\left[\pi_{it}(1,0) - \pi_{it}(0,0)\right] > \left[\pi_{it}(1,1) - \pi_{it}(0,1)\right] \tag{13}$$

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Under this assumption, they derive informative bounds around players' conditional choice probabilities when players are *level-k rational*, and show that the bounds become tighter as k increases. For instance, without further restrictions on beliefs (i.e., rationality of level 1), player i's conditional choice probability $P_{iT}(1)$ takes its largest possible value when $B_{iT}(1) = 0$, and it takes its smallest possible value when beliefs are $B_{iT}(0) = 1$. This result yields informative bounds on the period T choice probabilities of player i:

$$P_{iT}(1) \in [\Lambda_{iT} \left(\pi_{iT}(1,1) - \pi_{iT}(0,1) \right) \ , \ \Lambda \left(\pi_{iT}(1,0) - \pi_{iT}(0,0) \right)] \tag{14}$$

These bounds on conditional choice probabilities can be used to set identify the structural parameters in players' preferences.

In their setup, the monotonicity of players' payoffs in the decisions of other players implies monotonicity of players' best response probability functions (BRPF) in the beliefs about other players actions. This type of monotonicity is very convenient in their approach, not only from the perspective of identification, but also because it yields a very simple approach to calculate upper and lower bounds on conditional choice probabilities. However, this property does not extend to dynamic games, even the simpler ones. We now discuss this issue.

Consider the two-players, binary-choice, dynamic game at some period t smaller than T. To obtain bounds on players' choice probabilities analogous to the ones obtained at the last period, we need to find, for every value of the state variables $\mathbf{X}$, the smallest and largest feasible values of the best response $\Lambda_{it}(v_{it}^{\mathbf{B}}(1, \mathbf{X}) - v_{it}^{\mathbf{B}}(0, \mathbf{X}))$. That is, we need to minimize (and maximize) this best response with respect to beliefs $\{B_{it}, B_{it+1}, ..., B_{iT}\}$. Without making further assumptions, this best response function is not monotonic in beliefs at every possible state. In fact, this monotonicity is only achieved under very strong conditions not only on the payoff function but also on the transition probability of the state variables and on belief functions themselves.

Therefore, in a dynamic game, to find the largest and smallest value of a best response (and ultimately the bounds on choice probabilities) at periods $t < T$, one needs to explicitly solve a nontrivial optimization problem. In fact, the maximization (minimization) of the BRPF with respect to beliefs is a extremely complex task. The main reason is that the best response probability evaluated at a value of the state variables depends on beliefs at every period in the future and at every possible value of the state variables in the future. Therefore, to find bounds on best responses we must solve an optimization problem with a dimension equal to the number of values in the space of state variables times the number of future periods. This is because, in general, the maximization (minimization) of a best response with respect to beliefs does not have a time-recursive structure except under very special assumptions (see Aguirregabiria, 2008). For instance, though $B_{iT}(1|\mathbf{X}_T) = 0$ maximizes the best response at the last period T, in general the maximization of a best response at period $T-1$ is not achieved setting $B_{iT}(1|\mathbf{X}_T) = 0$ for any value of $\mathbf{X}_T$. More generally, the beliefs from period t to T that optimize best responses at t are not equal to the beliefs from period t to T that optimize best responses at $t-1$. So at each point in time we need to re-optimize with respect to beliefs about strategies at every period in the future. That is, while the optimization of expected and discounted payoffs has the well-known time-recursive structure,

the maximization (minimization) of the BRPFs does not.

In summary, the extension to dynamic games of the bounds approach, that Aradillas-Lopez and Tamer propose in the context of static, two-players, binary-choice games, suffers from substantial computational problems. Here we propose an alternative approach.

3 Identification

3.1 Conditions on Data Generating Process

Suppose that the researcher has panel data with realizations of the game over multiple geographic *locations* and time periods.[6] We use the letter m to index locations. The researcher observes a random sample of M locations with information on $\{Y_{imt}, \mathbf{X}_{mt}\}$ for every player $i \in \{1, 2, ..., N\}$ and every period $t \in \{1, 2, ..., T^{data}\}$. Note that T^{data} represents the number of periods in the data, while T is the time horizon of the dynamic game. If the game has a finite horizon ($T < \infty$), then we assume that the dataset includes all the periods in the game such that $T^{data} = T$. Obviously, for infinite horizon games we have that $T^{data} < T = \infty$. We assume that T^{data} is small and the number of local markets, M, is large. For the identification results in this section we assume that M is infinite. Since the main text deals with the finite horizon game, we use T for the rest of the paper to represent both the horizon of the game and the number of periods in the data. We consider the infinite horizon game in the Appendix. We assume that the only unobservable variables for the researcher are the private information shocks $\{\boldsymbol{\varepsilon}_{imt}\}$, which are assumed to be independently and identically distributed across players, markets, and over time.

We want to use this sample to estimate the structural 'parameters' or functions of the model: i.e., payoffs $\{\pi_{it}, \beta\}$; transition probabilities $\{f_t\}$; distribution of unobservables $\{\Lambda_{it}\}$; and beliefs $\{B_{it}\}$. For primitives other than players' beliefs, we make some assumptions that are standard in previous research on identification of static games and of dynamic structural models with rational or equilibrium beliefs.[7] We assume that the distribution of the unobservables, Λ_{it}, is known to the researcher up to a scale parameter. We study identification of the payoff functions π_{it} up to scale, but for notational convenience we omit the scale parameter.[8] Following the standard approach in dynamic decision models, we assume that the discount factor, β, is known to the researcher. Finally, note that the transition probability functions $\{f_t\}$ are nonparametrically identified.[9] Therefore, we concentrate on the identification of the payoff functions π_{it} and belief functions B_{it} and assume that $\{f_t, \Lambda_{it}, \beta\}$ are known.

Let $\mathbf{P}^0_{imt}$ be the vector of CCPs with the true (population) conditional probabilities $\Pr(Y_{imt} = y|i, m, t, \mathbf{X}_{mt} = \mathbf{X})$ for player i in market m at period t. Similarly, let $\mathbf{B}^0_{imt}$ be the vector of

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[6] In the context of empirical applications of games in Industrial Organization, a geographic location is a *local market*.

[7] See Bajari and Hong (2005), or Bajari et al (2010), among others.

[8] Aguirregabiria (2010) provides conditions for the nonparametric identification of the distribution of the unobservables in single-agent dynamic structural models. Those conditions can be applied to identify the distribution of the unobservables in our model.

[9] Note that $f_t(\mathbf{X}'|\mathbf{Y}, \mathbf{X}) = \Pr(\mathbf{X}_{mt+1} = \mathbf{X}' \mid \mathbf{Y}_{mt} = \mathbf{Y}, \mathbf{X}_{mt} = \mathbf{X})$. We can estimate consistently these conditional distributions using, for instance, kernel methods.

probabilities with the true values of player i's beliefs in market m at period t. And let $\boldsymbol{\pi}^0 \equiv \{\pi_{it}^0 : i = 1, 2; \, t = 1, 2, ..., T\}$ be the true payoff functions in the population. Assumption *ID-1* summarizes our conditions on the Data Generating Process.

ASSUMPTION ID-1. (A) For every player i, $\mathbf{P}_{imt}^0$ is the best response of player i given his beliefs $\mathbf{B}_{im}^0$ and the payoff functions $\boldsymbol{\pi}^0$. (B) A player has the same beliefs in two markets with the same observable characteristics $\mathbf{X}$, i.e., for every market m with $\mathbf{X}_{mt} = \mathbf{X}$, $B_{imt}(\mathbf{y}_{-i}|\mathbf{X}) = B_{it}(\mathbf{y}_{-i}|\mathbf{X})$.

Assumption *ID-1*(A) establishes that players are rational in the sense that their actual behavior is the best response given their beliefs. Assumption *ID-1*(B) establishes that a player has the same beliefs in two markets with the same state variables and at the same period of time. This assumption is common in the literature of estimation of games under the restriction of equilibrium beliefs (e.g., Bajari, Benkard, and Levin, 2007, or Bajari et al, 2010). Note that beliefs can vary across markets according to the state variables in $\mathbf{X}_{mt}$. This assumption allows players to have different belief functions in different markets as long as these markets have different values of time-invariant observable exogenous characteristics. For instance, beliefs could be a function of "market type," which are determined by some market specific time-invariant observable characteristics. If the number of market types is small (more precisely, if it does not increase with M), then we can allow players' beliefs to be completely different in each market type.[10]

In dynamic games where beliefs are in equilibrium, Assumption *ID-1* effectively allows the researcher to identify player beliefs. Under this assumption, conditional choice probabilities are identified, and if beliefs are in equilibrium, the belief of player i about the behavior of player j is equal to the conditional choice probability function of player j. When beliefs are not in equilibrium, Assumption *ID-1* is not sufficient for the identification of beliefs. However, assumption *ID-1* still implies that CCPs are identified from the data. This assumption implies that for any player i, any period t, and any value of $(y, \mathbf{X})$, we have that $P_{imt}^0(y|\mathbf{X}) = P_{it}^0(y|\mathbf{X}) = \Pr(Y_{imt} = y|\mathbf{X}_{mt} = \mathbf{X})$, and this conditional probability can be estimated consistently using the M observations of $\{Y_{imt}, \mathbf{X}_{mt}\}$ in our random sample of these variables. This in turn, as we will show, is important for the identification of beliefs themselves.

For notational simplicity, we omit the market subindex m for the rest of this section.

ASSUMPTION ID-2 ('Normalization' of payoff function): The one-period payoff function π_{it} is 'normalized' to zero for $Y_{it} = 0$, i.e., $\pi_{it}(0, \mathbf{Y}_{-it}, \mathbf{X}_t) = 0$ for any value of $(\mathbf{Y}_{-it}, \mathbf{X}_t)$.

Assumption *ID-2* establishes a 'normalization' of the payoff that is commonly adopted in many discrete choice models: the payoff to one of the choice alternatives, say alternative 0, is normalized to zero. The particular form of normalization of payoffs that we choose does not affect our identification results as long as the normalization imposes $A^{N-1}|\mathcal{X}|$ restrictions on each payoff function π_{it}.

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[10] It is also important to note that when we incorporate time-invariant unobserved market heterogeneity in our model we can allow for different belief functions for each market type, where now market types can be defined in terms of unobservables.

3.2 Identification of payoff and belief functions

In this subsection we examine different types of restrictions on payoffs and beliefs that can be used to identify dynamic games. The main point that we want to emphasize here is that restrictions that apply either only to beliefs or only to payoffs are not sufficient to identify this class of models. For instance, the assumption of equilibrium beliefs alone can identify beliefs but it is not enough to identify the payoff function. We also show that a exclusion restriction that has been commonly used to identify the payoff function can be exploited to relax the assumption of equilibrium beliefs.

Let $\mathbf{P}_{it}(\mathbf{X})$ be the $(A-1) \times 1$ vector of CCPs $(P_{it}(1|\mathbf{X}), ..., P_{it}(A-1|\mathbf{X}))$, and let $\widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X})$ be the $(A-1) \times 1$ vector of differential values $(\widetilde{v}_{it}(1, \mathbf{X}), ..., \widetilde{v}_{it}(A-1, \mathbf{X}))$. The model restrictions can be represented using the best response conditions $\mathbf{P}_{it}(\mathbf{X}) = \boldsymbol{\Lambda}\left(\widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X})\right)$, where $\boldsymbol{\Lambda}(\boldsymbol{v})$ is the vector-valued function $(\Lambda(1|\boldsymbol{v}), \Lambda(2|\boldsymbol{v}), ..., \Lambda(A-1|\boldsymbol{v}))$. Given these conditions, and our normalization assumption ID-2, we want to identify payoffs and beliefs. The distribution function Λ is invertible. Let $\mathbf{q}(\mathbf{P}) \equiv (q(1, \mathbf{P}), q(2, \mathbf{P}), ..., q(A-1, \mathbf{P}))$ be the inverse mapping of $\boldsymbol{\Lambda}$ such that if $\mathbf{P} = \boldsymbol{\Lambda}(\boldsymbol{v})$ then $\boldsymbol{v} = \mathbf{q}(\mathbf{P})$. Therefore, $\widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X}) = \mathbf{q}(\mathbf{P}_{it}(\mathbf{X}))$. For instance, for the multinomial logit case with $\Lambda(y|\boldsymbol{v}) = \exp\{v(y)\} / \sum_{y' \in \mathcal{Y}} \exp\{v(y')\}$, the inverse function $\mathbf{q}(\mathbf{P}_{it}(\mathbf{X}))$ is $q(y, \mathbf{P}_{it}(\mathbf{X})) = \ln(P_{it}(y|\mathbf{X})) - \ln(P_{it}(0|\mathbf{X}))$.

Given that CCPs are identified and that the distribution function Λ and the inverse mapping $\mathbf{q}(.)$ are known (up to scale) to the researcher, we have that the differential values $\widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X})$ are identified. Then, hereinafter, we treat $\widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X})$ as an identified object. To underline the identification of the value differences from inverting CCPs, we will often use $q(y, \mathbf{P}_{it}(\mathbf{X}))$, or with some abuse of notation $q_{it}(y, \mathbf{X})$, instead of $\widetilde{v}_{it}^{\mathbf{B}}(y, \mathbf{X})$.

The identification problem can be described in terms of the identification of payoffs π_{it} and beliefs B_{it} given differential values q_{it}. We can represent the relationship between differential values and payoffs and beliefs using a recursive system of linear equations. For every period t and $(y_i, \mathbf{X}) \in [\mathcal{Y} - \{0\}] \times \mathcal{X}$, the following equation holds:

$$q_{it}(y_i, \mathbf{X}) = \mathbf{B}_{it}(\mathbf{X})' \left[\boldsymbol{\pi}_{it}(y_i, \mathbf{X}) + \widetilde{\mathbf{c}}_{it}(y_i, \mathbf{X})\right] \tag{15}$$

where $\mathbf{B}_{it}(\mathbf{X})$, $\boldsymbol{\pi}_{it}(y_i, \mathbf{X})$, and $\widetilde{\mathbf{c}}_{it}(y_i, \mathbf{X})$ are vectors with dimension $A^{N-1} \times 1$. $\mathbf{B}_{it}(\mathbf{X})$ is the vector of beliefs $\{B_{it}(\mathbf{y}_{-i}|\mathbf{X}) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}\}$; $\boldsymbol{\pi}_{it}(y_i, \mathbf{X})$ is a vector of payoffs $\{\pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{X}) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}\}$; $\widetilde{\mathbf{c}}_{it}(y_i, \mathbf{X})$ is a vector of continuation value differences $\{c_{it}(y_i, \mathbf{y}_{-i}, \mathbf{X}) - c_{it}(0, \mathbf{y}_{-i}, \mathbf{X}) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}\}$, and $c_{it}(\mathbf{Y}_t, \mathbf{X}_t)$ is the *continuation value function* that provides the expected and discounted value of *future* payoffs given future beliefs, current state, and current choices of *all* players:

$$c_{it}(\mathbf{Y}_t, \mathbf{X}_t) \equiv \beta \int V_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1}, \boldsymbol{\varepsilon}_{it+1}) \, dG_{it}(\boldsymbol{\varepsilon}_{it+1}) \, f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t) \tag{16}$$

By definition, continuation values at the last period T are zero, $c_{iT}(\mathbf{Y}, \mathbf{X}) = 0$.

The system of equations (15) summarizes all the restrictions of the model. These systems of equations have a recursive nature such that the continuation values in $\widetilde{\mathbf{c}}_{it}(y_i, \mathbf{X})$ are determined by payoffs at periods greater than t. Therefore, following a backwards induction argument, for every player i and period t we have $(A-1)|\mathcal{X}|$ restrictions (i.e., as many restrictions as there are free

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values $q_{it}(y_i, \mathbf{X})$), and the number of unknowns is $(A-1)A^{N-1}|\mathcal{X}|$ in the payoff function π_{it}, and $(A^{N-1}-1)|\mathcal{X}|$ in the beliefs function B_{it}.

Table 2 presents the number of parameters, restrictions, and over- or under- identifying restrictions under different conditions on the model. The table presents these numbers as ratios with respect to the total number of possible free actions and states per player, i.e., we divide number of parameters and restrictions by $(A-1)|\mathcal{X}|$. The first row in table 2 presents the case with completely unrestricted beliefs and payoffs. The best response conditions imply $(A-1)|\mathcal{X}|$ restrictions, or 1 restriction for each free value of $(y_i, \mathbf{X})$. However, the model has as many as $(A-1)A^{N-1}|\mathcal{X}|$ parameters in the payoff function $\pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t)$, and $(A^{N-1}-1)|\mathcal{X}|$ unknowns in the beliefs functions $B_{it}(\mathbf{y}_{-i}|\mathbf{X})$. It is simple to verify that the order condition for identification is not satisfied.

The second row in table 2 presents the case under the assumption of equilibrium beliefs but unrestricted payoff function. The equilibrium beliefs assumption implies $(A^{N-1}-1)|\mathcal{X}|$ additional restrictions, i.e., $B_{it}(\mathbf{y}_{-i}|\mathbf{X}) = \prod_{j\neq i} P_{jt}(y_j|\mathbf{X})$ for every free value of $(\mathbf{y}_{-i}, \mathbf{X})$. It is obvious that these additional restrictions identify beliefs. However, they are not enough to identify the payoff function. Therefore, even if a researcher is willing to assume equilibrium beliefs, he still has to impose restrictions on the payoff function in order to get identification.

Assumption *ID-3* presents nonparametric restrictions on the payoff function that, combined with the assumption of equilibrium beliefs, are typically used for identification in games with equilibrium beliefs.[11]

ASSUMPTION ID-3 (Exclusion Restriction): (i) The vector of state variables $\mathbf{X}_t$ can be partitioned in two subvectors, $\mathbf{X}_t = (\mathbf{S}_t, \mathbf{W}_t)$, such that the vector $\mathbf{W}_t \in \mathcal{W}$ includes variables that enter in the payoff function of more than one player (or even all the players), and $\mathbf{S}_t = (S_{1t}, S_{2t}, ..., S_{Nt}) \in \mathcal{S}^N$ where S_{it} represents state variables that enter into the payoff function of player i but not the payoff function of any of the other players. Therefore, the payoff function π_{it} depends on $(S_{it}, \mathbf{W}_t)$ but not on $\mathbf{S}_{-it}$,

$$\pi_{it}(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{S}_{-it}, \mathbf{W}_t) = \pi_{it}(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{S}'_{-it}, \mathbf{W}_t) \quad \text{for any } \mathbf{S}'_{-it} \neq \mathbf{S}_{-it} \quad (17)$$

(ii) The number of states in $\mathcal{S}$ is greater or equal than the number of actions A, i.e., $|\mathcal{S}| \geq A$. (iii) Conditional on $(\mathbf{S}_{-it}, \mathbf{W}_t)$, the probability distribution of S_{it} has positive probability at every point in its support $\mathcal{S}$.

With some abuse of notation we use $\pi_{it}(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{W}_t)$, instead of $\pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t)$, to represent the payoff function under assumption ID-3. Furthermore, the vector of common state variables $\mathbf{W}_t$ does not play any role in the identification of the model, and then we will omit in some of our expressions.

The exclusion restriction in assumption *ID-3* is common in empirical applications of dynamic games.

EXAMPLE 2: Consider the *dynamic game of market entry and exit* that we introduced in Example 1. The vector of common knowledge state variables of the game is $\mathbf{X}_t = (H_t, Z_i, Y_{it-1} : i =$

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[11]See Aguirregabiria and Mira (2002), Pesendorfer and Schmidt-Dengler (2003), Bajari and Hong (2005), Bajari, Hong, and Ryan (2010), and Bajari et al. (2011), among others.

$1, 2, ..., N$). The specification of the model implies that market size H_t enters in the payoff of every firm. However, a firm's own incumbency status at previous period, Y_{it-1}, and the time-invariant characteristic affecting its fixed cost, Z_i, enter only into the profit function of firm i but not in the profits of the other firms. Therefore, in this example, $S_{it} = (Z_i, Y_{it-1})$ and $\mathbf{W}_t = H_t$. ∎

The third row in table 2 presents the case when we impose equilibrium restrictions on beliefs and the exclusion restriction on payoffs (assumption ID-3). Under assumption ID-3, the state space $\mathcal{X}$ is equal to $\mathcal{W} \times \mathcal{S}^N$, and the ratio between the number of parameters in the payoff function and the total number of actions-states is equal to $(A/|\mathcal{S}|)^{N-1}$. Then, it is simple to verify that the order condition of identification is satisfied if the number of points in the space of the *special variable(s)* in the exclusion restriction, $|\mathcal{S}|$, is greater or equal than the number of choice alternatives A, i.e., condition (ii) in assumption ID-3. The rank condition for identification is satisfied under the condition of full support variation of S_{it} conditional on $(\mathbf{S}_{-it}, \mathbf{W}_t)$, i.e., condition (iii) in assumption ID-3. Therefore, equilibrium beliefs and a exclusion restriction in payoffs can fully identify dynamic games. In fact, when the number of states in the set $\mathcal{S}$ is strictly greater than the number of possible actions, the restrictions implied by equilibrium conditions overidentify payoffs. That is the case in the game in Example 1. The dimension of the space of $S_{it} = (Z_i, Y_{it-1})$ is $|\mathcal{Z}|A$ that is greater than the number of actions.

The fourth row in Table 2 shows that the exclusion restriction alone, without any restriction on beliefs, is not enough to identify the model.

The following assumption presents a restriction on beliefs that is weaker than the assumption of equilibrium beliefs and that together with assumptions ID-1 to ID-3 is sufficient to nonparametrically identify payoffs and beliefs in the model.

ASSUMPTION ID-4: Let $\mathcal{S}^{(R)} \subset \mathcal{S}$ be a subset of values in the set $\mathcal{S}$, with dimension $|\mathcal{S}^{(R)}| \equiv R$ that is greater or equal than A and strictly smaller than $|\mathcal{S}|$.

> *(a) For every state $\mathbf{X} = (S_i, \mathbf{S}_{-i}, \mathbf{W})$ such that $\mathbf{S}_{-i} \in [\mathcal{S}^{(R)}]^{N-1}$, the beliefs of player i are such that $B_{it}(\mathbf{y}_{-i}|\mathbf{X}) = P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ where $P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ represents either the actual conditional choice probabilities of the other players, $\prod_{j \neq i} P_{jt}(y_j|\mathbf{X})$, or consistent estimates of beliefs based on elicited beliefs data.*
> *(b) Let $\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})$ be the $R^{N-1} \times A^{N-1}$ matrix with elements $\{P_{-it}(\mathbf{y}_{-i}|S_i, \mathbf{S}_{-i}, \mathbf{W}) : \mathbf{y}_{-i} \in \mathcal{Y}^{\mathbf{N-1}}, \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$. For every period t and any value of $(S_i, \mathbf{W})$, this matrix has rank A^{N-1}.*

Condition (a) establishes that there are some values of the opponents' stock variables $\mathbf{S}_{-i}$ for which strategic uncertainty disappears and beliefs about opponents' choice probabilities become unbiased. Alternatively, this assumption could be motivated by the availability of data on elicited beliefs for a limited number of states. Since $\mathcal{S}^{(R)}$ is a subset of the space $\mathcal{S}$, it is clear that Assumption ID-4(a) is weaker than the assumption of equilibrium beliefs, or alternatively, it is weaker than the condition of observing elicited beliefs for every possible value of the state variables.

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Condition (b) is needed for the rank condition of identification. A stronger but more intuitive condition than (b) is that $P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ is strictly monotonic with respect to $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, the actual choice probabilities of the other players depend monotonically on the state variables in $\mathbf{S}_{-i}$.

EXAMPLE 3: For the dynamic game in our example, the space $\mathcal{S}$ is equal to $\mathcal{Z} \times \mathcal{Y}$, with $\mathcal{Z}$ being the space of Z_i and $\mathcal{Y}$ is the binary set $\{0,1\}$. Suppose that set $\mathcal{S}^{(R)}$ consists of a pair of values $\{Z^*, 0\}$ and $\{Z^*, 1\}$, where Z^* is a particular point in the support $\mathcal{Z}$. Assumption ID-4 establishes that for every value of (H_t, Z_i, Y_{it-1}) we have that:

$$B_{it}(1|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 0) \quad = \quad P_{jt}(1|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 0)$$

$$B_{it}(1|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 1) \quad = \quad P_{jt}(1|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 1)$$

That is, when the value of Z_j is Z^*, player i has unbiased beliefs about the behavior of player j whatever is the value of $(H_t, Z_i, Y_{it-1}, Y_{jt-1})$. In this example, $\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})$ is the 2×2 matrix:

$$\mathbf{P}_{jt}^{(R)}(H_t, Z_i, Y_{it-1}) = \left[\begin{array}{ll} P_{jt}(0|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 0) \,, & P_{jt}(1|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 0) \\ P_{jt}(0|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 1) \,, & P_{jt}(1|H_t, Z_i, Y_{it-1}, Z_j = Z^*, Y_{jt-1} = 1) \end{array} \right]$$

Condition (b) on the rank of $\mathbf{P}_{jt}^{(R)}$ is satisfied if $P_{jt}(1|., Y_{jt-1} = 0) \neq P_{jt}(1|., Y_{jt-1} = 1)$, i.e., if being an incumbent in the market at previous period has a non-zero effect on the probability of being in the market at current period. This is a very weak condition that we expect to be always satisfied in a dynamic game of market entry and exit. ∎

In the last row of table 2 we present the order condition of identification under assumptions ID-3 and ID-4. This condition is satisfied if $1 - \dfrac{A^{N-1}}{|S|^{N-1}} + \left(\dfrac{R^{N-1}}{|S|^{N-1}} - 1 \right) \dfrac{A^{N-1} - 1}{A - 1}$ is greater or equal than zero. When the number of players in the game is two, this expression becomes $\dfrac{R}{|S|} - \dfrac{A}{|S|}$, and therefore the order condition of identification is satisfied if $R \geq A$, which is one of the conditions in assumption ID-4. However, with more than two players in the game, we have that $R \geq A$ is not enough to guarantee the order condition.

Proposition 1 formalizes our main identification result.

PROPOSITION 1: Suppose that assumptions ID-1 to ID-4 hold, and that the sequence of payoff functions and payoffs functions between periods $t + 1$ and T, $\{\pi_{it'}, B_{it'} : t' = t + 1, ..., T\}$, are identified. Then:

> *(i) The payoff function at period t, $\pi_{it}(y_i, \mathbf{y}_{-i}, S_i, \mathbf{W})$, is identified;*
> *(ii) If $N = 2$ and matrix $\mathbf{Q}_{it}^{(R)}(S_i, \mathbf{W})$, with dimension $A \times R^{N-1}$ and elements $\{q_{it}(y_i, S_i, \mathbf{S}_{-i}, \mathbf{W}) : y_i \in \mathcal{Y}, \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$, has rank equal to A, then the beliefs function $B_{it}(\mathbf{y}_{-i}|\mathbf{X})$ is identified.* ∎

Proof. In the Appendix.

COROLLARY: Suppose that: (1) assumptions ID-1 to ID-4 hold; (2) $N = 2$; and (3) for every period t and every value of $(S_i, \mathbf{W})$, matrix $\mathbf{Q}_{it}^{(R)}(S_i, \mathbf{W})$ has rank equal to A. Then, payoff

functions and beliefs functions are nonparametrically identified everywhere and at every period t.
∎

Remark 1. The condition that the rank of $\mathbf{Q}_{it}^{(R)}(S_i, \mathbf{W})$ is equal to A, in condition (ii), is satisfied if the conditional choice probability function of player i is strictly monotonic in $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, the actual choice probabilities of the other players depend monotonically on the state variables in $\mathbf{S}_{-i}$. Note that for the identification of the payoff function we need that beliefs (or the choice probabilities of players other than i) depend monotonically on $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. And for the identification of beliefs we also need that the choice probability of the own player i depends on $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, to identify beliefs we need that player i is playing a game such that the values of the state variables of the other players affect his decision through the effect of these variables in they beliefs. If the other players' actions do not have any effect on the payoff of player i, then his beliefs do not have any effect on his actions and therefore his actions cannot reveal any information about his beliefs.

Remark 2. The non-identification of beliefs with more than two players may seem a very negative result. However, it is quite intuitive given the high dimensionality of beliefs functions relative to the dimensionality of the CCP function of a single player. Of course, there are different types of "nonparametric" restrictions on beliefs that can provide identification with more than two players. Perhaps, the most obvious restriction is that player i believes that all the other players behave in the same way. Under this condition, the probability distribution $B_{it}(\mathbf{y}_{-i}|\mathbf{X})$ over $\mathcal{Y}^{N-1}$ can be characterized in terms of only $A-1$ free probabilities, regardless the number of players N, and identification can be achieved. A different approach to achieve identification with more than two players is to impose restrictions of homogeneity of beliefs across players. For instance, with three players, we may assume that any pair of players (i, j) have the same beliefs about the behavior of the other player. Then, using the CCPs of the three players simultaneously, it is possible to identify beliefs of all the players. Of course, this type of restrictions may be plausible only in some particular applications. For the rest of the paper we concentrate in dynamic games with two players. We use subindexes i and j to represent the two players.

Our proof of Proposition 1 is constructive and it provides closed-form expressions of the unknown parameters (payoffs and beliefs) in terms of the identified CCP functions. We use these formulas to construct two-step nonparametric estimators of payoffs and beliefs that we describe in section 4. Interestingly, the expressions describing the identification of payoffs and beliefs have an interpretation as OLS estimators. This interpretation is useful not only for the actual implementation of the estimator and for the derivation of asymptotic properties but also to understand the conditions under which identification can be weaker or stronger.

Let $\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W})$ and $\mathbf{B}_{it}(\mathbf{X})$ be the $A \times 1$ vectors with payoffs $\pi_{it}(y_i, y_j, S_i, \mathbf{W})$ and beliefs $B_{it}(y_j|\mathbf{X})$, respectively, for every $y_j \in \mathcal{Y}$. At any period t and for any possible value $(y_i, S_i, \mathbf{W})$, the vector of payoffs $\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W})$ is identified as:

$$\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W}) = \left[\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \, \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W}) \right]^{-1} \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \, \tilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i, \mathbf{W}) \qquad (18)$$

where the $R \times A$ matrix $\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})$ has been defined in Assumption ID-4, and $\tilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i, \mathbf{W})$ is the $R \times 1$ vector with elements $\{\tilde{q}_{it}(y_i, S_i, \mathbf{S}_{-i}, \mathbf{W}) : \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$ with

$$\tilde{q}_{it}(y_i, \mathbf{X}) \equiv q_{it}(y_i, \mathbf{X}) - \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} P_{-it}(\mathbf{y}_{-i}|\mathbf{X}) \, c_{it}(y_i, \mathbf{y}_{-i}, \mathbf{X}) \tag{19}$$

$\tilde{\mathbf{q}}_{it}^{(R)}$ depends on the continuation values c_{it}. At the last period T these continuation values are zero. For any period t before the last period, continuation values can be obtained using a simple recursive formula that we present in Appendix 1. Note that the right-hand-side of equation (18), that describes the identification of the payoff function, is the OLS estimator from a linear regression of $\tilde{\mathbf{q}}_{it}^{(R)}$ on $\mathbf{P}_{-iT}^{(R)}$. This interpretation has an important and useful implication for the choice of the subset $\mathcal{S}_{-i}^{(R)}$: the greater the variability of P_{it} over the set $\mathcal{S}_{-i}^{(R)}$, the more precise the estimation of the payoff function.

At any period t and for any value of the vector of state variables $\mathbf{X}$, the $A \times 1$ vector of beliefs $\mathbf{B}_{it}(\mathbf{X})$ is identified as:

$$\mathbf{B}_{it}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X}) \tag{20}$$

$\mathbf{q}_{it}(\mathbf{X})$ is an $A \times 1$ vector with elements $\{q_{it}(1, \mathbf{X}), ..., q_{it}(A-1, \mathbf{X})\}$ at rows 1 to $A-1$, and a 1 at the last row. And $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is an $A \times A$ matrix where the element $(y_i, y_j + 1)$ is:

$$\pi_{it}(y_i, y_j, S_i, \mathbf{W}) + [c_{it}(y_i, y_j, \mathbf{X}) - c_{it}(0, y_j, \mathbf{X})] \tag{21}$$

and the last row of the matrix is a row of ones. By construction, the expression $\left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X})$ is equal to $\mathbf{P}_{-it}(\mathbf{X})$ for values of $\mathbf{X}$ such that $\mathbf{S}_{-i}$ belongs to $\mathcal{S}_i^{(R)}$. As in the case of the identification of payoffs, matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ depends on continuation values that in turn depend on future payoffs and beliefs. Using backwards induction we can obtain these continuation values recursively.

3.3   Test of unbiased beliefs

Though Assumptions ID-1 to ID-3 are not sufficient for the identification of payoffs and beliefs, they provide enough restrictions to test the null hypothesis of unbiased beliefs when the game has two players.

There are $N = 2$ players, i and j, the vector of state variables $\mathbf{X}$ is $(S_i, S_j, \mathbf{W})$, and players' actions are y_i and y_j. Let s_j^0 be an arbitrary value of in the set $\mathcal{S}$. And let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be two different subsets included in the set $\mathcal{S} - \{s_j^0\}$ such that they satisfy two conditions: (1) each of the sets has $A - 1$ elements; and (2) $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have at least one element that is different. Since $|\mathcal{S}| \geq A + 1$, it is always possible to construct two subsets that satisfy these conditions. Given these sets, we can define the $(A-1) \times (A-1)$ matrices of beliefs $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W})$ and $\Delta\mathbf{B}_{it}^{(b)}(S_i, \mathbf{W})$, where $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W})$ has elements $\{B_{it}(y_j, S_i, S_j, \mathbf{W}) - B_{it}(y_j, S_i, s_j^0, \mathbf{W}) : \text{for } y_j \in \mathcal{Y} - \{0\} \text{ and } S_j \in \mathcal{S}^{(a)}\}$, and $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W})$ has the same definition but for subset $\mathcal{S}^{(b)}$. Similarly, we can define matrices $\Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W})$ and $\Delta\mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})$, with elements $\{q_{it}(y_i, S_i, S_j, \mathbf{W}) - q_{it}(y_i, S_i, s_j^0, \mathbf{W}) : \text{for } y_i \in \mathcal{Y} - \{0\} \text{ and } S_j \in \mathcal{S}^{(a)}\}$, and matrices $\Delta\mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W})$ and $\Delta\mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})$, with elements $\{P_{jt}(y_j|S_i, S_j, \mathbf{W}) - P_{jt}(y_j|S_i, s_j^0, \mathbf{W}) : \text{for } y_j \in \mathcal{Y} - \{0\} \text{ and } S_j \in \mathcal{S}^{(a)}\}$.

PROPOSITION 2: Suppose that assumptions ID-1 and ID-3 hold and the model is such that $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t) = f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{W}_t)$. *Then:*

(a) The $(A-1) \times (A-1)$ *matrix of beliefs* $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{B}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1}$ *is identified from the CCPs of player* i *as* $\Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1}$;

(b) Under the assumption of unbiased beliefs, $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{B}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1}$ *is also identified from the CCPs of the other player,* j, *as* $\Delta\mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})\right]^{-1}$;

(c) Combining (a) and (b), the assumption of unbiased beliefs for player i *implies the following* $(A-1)^2$ *restrictions between CCPs of players* i *and* j:

$$\Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})\right]^{-1} = \mathbf{0} \qquad \blacksquare$$

Proof. In the Appendix.

Under the conditions of Proposition 2, for every value of $(S_i, \mathbf{W})$ there are $(A-1)^2$ functions, or objects, that depend only on the beliefs of player i and not on payoffs such that it is possible to identify these functions using choice probabilities of player i, i.e., revealed preference identifies these functions of beliefs. Of course, if we assume that beliefs are unbiased, we know that these beliefs are equal to the choice probabilities of the other player, and therefore we have a completely different form, with different data, to identify these functions of beliefs. If the hypothesis of equilibrium beliefs is correct, then both approaches should give us the same result. Therefore, the restriction provides a natural approach to test for the null hypothesis of equilibrium or unbiased beliefs.

EXAMPLE 4: Suppose that the dynamic game has two players making binary choices: $N = 2$ and $A = 2$. Then, subsets $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have only one element each: $\mathcal{S}^{(a)} = \{s^{(a)}\}$ and $\mathcal{S}^{(b)} = \{s^{(b)}\}$ with $s^{(a)} \neq s^0$, $s^{(b)} \neq s^0$, and $s^{(a)} \neq s^{(b)}$. By Proposition 2, for a given selection of $(s^0, s^{(a)}, s^{(b)})$, and a given value of $(S_i, \mathbf{W})$, the hypothesis of unbiased beliefs implies one testable restriction. The restriction has this form:

$$\frac{q_{it}(1, S_i, s^{(a)}, \mathbf{W}) - q_{it}(1, S_i, s^0, \mathbf{W})}{q_{it}(1, S_i, s^{(b)}, \mathbf{W}) - q_{it}(1, S_i, s^0, \mathbf{W})} - \frac{P_{jt}(1|S_i, s^{(a)}, \mathbf{W}) - P_{jt}(1|S_i, s^0, \mathbf{W})}{P_{jt}(1|S_i, s^{(b)}, \mathbf{W}) - P_{jt}(1|S_i, s^0, \mathbf{W})} = 0 \qquad (22)$$

It is clear that we can estimate nonparametrically all the components of this expression and implement a test. In section 4, we describe a test of the null hypothesis of unbiased beliefs based on this result. $\blacksquare$

4   Estimation and Inference

Our proof of identification above suggests a method for the estimation of the nonparametric model. Section 4.1 provides a description of that estimation method. In most empirical applications, the payoff function is parametrically specified. For this reason, in section 4.2 we extend the estimation method to deal with parametric models. Section 4.3 presents our test for the null hypothesis of unbiased beliefs.

4.1 Estimation with nonparametric payoff function

Nonparametric estimation proceeds in two steps.

Step 1: Nonparametric estimation of CCPs and transition probabilities. For every player, time period, and state $\mathbf{X}$, we estimate CCPs $P_{it}(y_i|\mathbf{X})$, and (if necessary) the transition probabilities $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$. We also construct estimates of $q_{it}(y_i, \mathbf{X})$ by inverting the distribution Λ, i.e., $q_{it}(y_i, \mathbf{X}) = \Lambda^{-1}(y_i, \mathbf{P}_{it}(\mathbf{X}))$.

Step 2: Recursive estimation of preferences and beliefs. We select the subset $\mathcal{S}^{(R)}$ with the values of S_j for which we assume that player i's beliefs are unbiased. Given this set and the estimates in step 1, we construct, for every period t and any value of $(y_i, S_i, \mathbf{W})$, the matrices $\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})$ and the vectors $\mathbf{q}_{it}(y_i, S_i, \mathbf{W})$. Then, we apply the following recursive procedure. We start at the last period T, where the continuation function is zero, and apply recursively the following three steps to estimate payoffs, beliefs, and continuation values functions at every period t: (i) the payoff function,

$$\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W}) = \left[\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \, \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})\right]^{-1} \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \, \tilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i, \mathbf{W}) \; ; \tag{23}$$

(ii) the beliefs function,

$$\mathbf{B}_{it}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X}) \; ; \tag{24}$$

and (iii) integrated value function,

$$c_{it}(y_i, y_j, \mathbf{X}) \;\; = \;\; \beta \sum_{\mathbf{X}_{t+1}} \ln \left(\sum_{y_{it+1}} \exp\left\{\mathbf{B}_{it}(\mathbf{X})' \left[\boldsymbol{\pi}_{it}(y_{it+1}, \mathbf{X}) + \mathbf{c}_{it}(y_{it+1}, \mathbf{X})\right]\right\} \right) f_t(\mathbf{X}_{t+1}|y_i, y_j, \mathbf{X}) \tag{25}$$

This estimator is consistent and asymptotically normal. The derivation of the asymptotic variance is cumbersome. In our empirical application we use the bootstrap method to obtain standard errors and confidence intervals for the estimates.

4.2 Estimation with parametric payoff function

In most applications the researcher assumes a parametric specification of the payoff function. A very common class of parametric specifications is the linear in parameters model:

$$\pi_{it}(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) = g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) \, \boldsymbol{\theta}_{it} \tag{26}$$

where $g(Y_i, Y_j, S_i, \mathbf{W})$ is a $1 \times K$ vector of known functions, and $\boldsymbol{\theta}_{it}$ is a $K \times 1$ vector of unknown structural parameters in player i's payoff function. Let $\boldsymbol{\theta}_i$ be the vector with all the parameters in the payoff of player i: $\boldsymbol{\theta}_i \equiv \{\boldsymbol{\theta}_{it} : t = 1, 2, ..., T\}$.

EXAMPLE 5: Consider the dynamic game in Example 1. The profit function in equation (2) can be written as $g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) \, \boldsymbol{\theta}_i$, where the vector of parameters $\boldsymbol{\theta}_i$ is $(\theta_i^M, \theta_i^D, \theta_{i0}^{FC}, \theta_{i1}^{FC}, \theta_i^{EC})'$ and

$$g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) = Y_{it} \{ \; H_t, \; -H_t Y_{jt}, \; -1, \; -Z_i, \; -1\{Y_{it-1} = 0\} \; \} \tag{27}$$

To estimate $\boldsymbol{\theta}_i$ we propose a simple three steps method. The first two-steps are the same as for the nonparametric model.

Step 3: Given the estimates from step 2, we can apply a pseudo maximum likelihood method in the spirit of Aguirregabiria and Mira (2002, 2007) to estimate the structural parameters $\boldsymbol{\theta}_i$. Define the following pseudo likelihood function for the model i.i.d. extreme value $\varepsilon's$:

$$Q(\boldsymbol{\theta}_i, \mathbf{B}_i, \mathbf{P}_i) \equiv \sum_{m=1}^{M} \sum_{t=1}^{T} \log \left(\frac{\exp\left\{ \widetilde{g}_{it}^{\mathbf{B},\mathbf{P}}(Y_{imt}, \mathbf{X}_{mt}) \, \boldsymbol{\theta}_i + \widetilde{e}_{it}^{\mathbf{B},\mathbf{P}}(Y_{imt}, \mathbf{X}_{mt}) \right\}}{\sum_{y_i=0}^{A-1} \exp\left\{ \widetilde{g}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X}_{mt}) \, \boldsymbol{\theta}_i + \widetilde{e}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X}_{mt}) \right\}} \right) \tag{28}$$

$\widetilde{g}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X})$ is the discounted sum of the expected values of $\{g(Y_{jt+s}Y_{jt+s}, \mathbf{X}_{t+s}) : s = 0, 1, ..., T-t\}$ given that the state at period t is $\mathbf{X}$, that player i chooses alternative y_i at period t and then behaves according to the choice probabilities in $\mathbf{P}$, and believes that player j behaves according to the probabilities in $\mathbf{B}$. And $\widetilde{e}_{imt}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X})$ is also a discounted sum, but of expected future values of $\sum_{y_i=0}^{A-1} P_{it+s}(y_i|\mathbf{X}_{mt+s}) [\gamma - \ln P_{it+s}(y_i|\mathbf{X}_{mt+s})]$, that represents the expected value of $\varepsilon_{im,t+s}(Y_{imt+s})$ when $Y_{im,t+s}$ is optimally chosen, and γ is Euler's constant. From steps 1 and 2, we have consistent estimates of CCPs, $\hat{\mathbf{P}}_i$, and beliefs, $\hat{\mathbf{B}}_i$. Then, a consistent pseudo maximum likelihood estimator of $\boldsymbol{\theta}_i$ is defined as the value $\hat{\boldsymbol{\theta}}_i^{(1)}$ that maximizes $Q(\boldsymbol{\theta}_i, \hat{\mathbf{B}}_i, \hat{\mathbf{P}}_i)$. Note that the sample criterion function $Q(\boldsymbol{\theta}_i, \hat{\mathbf{B}}_i, \hat{\mathbf{P}}_i)$ is just the log likelihood function of a Conditional Logit model with the restriction that the parameter multiplying the discounted sum $\widetilde{e}_{it}^{\mathbf{B},\mathbf{P}}$ is equal to 1. The estimator is root-M consistent and asymptotically normal.

Steps 1 to 3 can be applied recursively to improve the statistical properties of our estimators. Let $\hat{\mathbf{P}}_i^{(1)} = \{\hat{P}_{it}^{(1)}(y_i|\mathbf{X})\}$ be the vector with the new estimates of CCPs implied by the parametric model:

$$\hat{P}_{it}^{(1)}(y_i|\mathbf{X}) = \frac{\exp\left\{ \widetilde{g}_{it}^{\hat{\mathbf{B}}^0, \hat{\mathbf{P}}^0}(y_i, \mathbf{X}) \, \widehat{\boldsymbol{\theta}}_i^{(1)} + \widetilde{e}_{it}^{\hat{\mathbf{B}}^0, \hat{\mathbf{P}}^0}(y_i, \mathbf{X}) \right\}}{\sum_{y_i'=0}^{A-1} \exp\left\{ \widetilde{g}_{it}^{\hat{\mathbf{B}}^0, \hat{\mathbf{P}}^0}(y_i', \mathbf{X}) \, \widehat{\boldsymbol{\theta}}_i^{(1)} + \widetilde{e}_{it}^{\hat{\mathbf{B}}^0, \hat{\mathbf{P}}^0}(y_i', \mathbf{X}) \right\}} \tag{29}$$

where $\hat{\mathbf{B}}^0$ and $\hat{\mathbf{P}}^0$ represent the initial nonparametric estimators of beliefs and CCPs, respectively, and $\hat{\boldsymbol{\theta}}_i^{(1)}$ is the pseudo maximum likelihood estimator of the parameters in the payoff function. We expect $\hat{\mathbf{P}}_i^{(1)}$ to have smaller asymptotic variance and finite sample bias than the initial $\hat{\mathbf{P}}_i^{(0)}$. Given $\hat{\mathbf{P}}_i^{(1)}$ and parametric estimates of payoffs, $g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) \, \hat{\boldsymbol{\theta}}_{it}^{(1)}$, we can construct new estimates of beliefs using the formula in equation (24). That is:

$$\widehat{\mathbf{B}}_{it}^{(1)}(\mathbf{X}) = \left[\widehat{\mathbf{V}}_{it}^{(1)}(\mathbf{X}) \right]^{-1} \widehat{\mathbf{q}}_{it}^{(1)}(\mathbf{X}) \, , \tag{30}$$

where $\widehat{\mathbf{V}}_{it}^{(1)}$ and $\widehat{\mathbf{q}}_{it}^{(1)}$ are constructed using $\hat{\mathbf{P}}_i^{(1)}$ and $\hat{\boldsymbol{\theta}}_i^{(1)}$. We can apply this procedure recursively to update CCPs, beliefs, and structural parameters and to obtain a sequence of estimators $\{\hat{\boldsymbol{\theta}}^{(K)}, \hat{\mathbf{B}}^{(K)}, \hat{\mathbf{P}}^{(K)} : K \geq 1\}$.

4.3 Test of equilibrium beliefs

In principle, we could use a standard Lagrange Multiplier (LM) or Score test of the null hypothesis of equilibrium beliefs. That test is based on the constrained maximum likelihood estimation (MLE) of structural parameters and beliefs.[12] In our context, tht LM test has at least two important limitations. First, maximum likelihood estimation of dynamic games is computationally very demanding both because the high dimension of the state space and because of the existence of multiple equilibria. Second, this is a general specification test. The null hypothesis is not only that beliefs are in equilibrium but also that the parametric specification of preferences and the distribution of unobservables is correct. We would like to have a procedure that specifically tests for the equilibrium beliefs and not for other specification assumptions of the model.

Our test is based on Proposition 2. Here we describe our test of unbiased beliefs at a single value of the state $(S_i, \mathbf{W})$. However, it is straightforward to generalize the test to multiple states or every possible state. Let $\delta_i(S_i, \mathbf{W})$ the be $(A-1)^2 \times 1$ vector with the elements of matrix $\Delta \mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W})$ $\left[\Delta \mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1} - \Delta \mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W}) \left[\Delta \mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})\right]^{-1}$. And let $\widehat{\delta}_i(S_i, \mathbf{W})$ be a consistent estimator of this vector. Define the statistic:

$$\hat{D} = \widehat{\delta}_i(S_i, \mathbf{W}) \left[\widehat{Var}\left(\widehat{\delta}_i(S_i, \mathbf{W})\right)\right]^{-1} \widehat{\delta}_i(S_i, \mathbf{W}) \tag{31}$$

where $\widehat{Var}\left(\widehat{\delta}_i(S_i, \mathbf{W})\right)$ is a consistent estimator of the asymptotic variance of $\widehat{\delta}_i(S_i, \mathbf{W})$, that can be obtained using a nonparametric bootstrap. Under the null hypothesis of unbiased beliefs, the statistic $\hat{D}$ is asymptotically distributed as a Chi-square with $(A-1)^2$ degrees of freedom. Under the alternative hypothesis, beliefs are biased, $\widehat{\delta}_i(S_i, \mathbf{W})$ does not converge in probability to zero, and the statistic $\hat{D}$ has a non-central chi-square asymptotic distribution.

5 Monte Carlo Experiments

In this section we use Monte Carlo methods to illustrate the identification and estimation framework presented in previous sections. Our purpose in implementing Monte Carlo experiments is threefold. First, we would like to assess the power of our identification assumptions in small samples. While preferences and beliefs are asymptotically identified given our identification assumptions, it might well be the case that when we replace the assumption of equilibrium beliefs by our identification conditions, estimates of preferences and beliefs become imprecise in the small samples that are common in actual applications. We want to evaluate the price, in terms of precision of our estimates, that we have to pay for relaxing the assumption of unbiased beliefs. The second purpose of our Monte Carlo experiments is to study the consequences of imposing the assumption of equilibrium beliefs when this assumption does not hold in the data generating process. In principle, incorrectly

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[12]Define the log-likelihood function, $l(\boldsymbol{\theta}, \mathbf{P}) \equiv \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{N} \log \Lambda(Y_{imt}, v_{it}^{\mathbf{P}}(\mathbf{X}_{mt}, \boldsymbol{\theta}))$. The constrained MLE is defined as a vector $(\hat{\boldsymbol{\theta}}_{MLE}, \hat{\mathbf{P}}_{MLE})$ that maximizes the likelihood $l(\boldsymbol{\theta}, \mathbf{P})$ subject to the equilibrium constraints $\mathbf{P} = \Lambda(v^{\mathbf{P}}(\boldsymbol{\theta}))$. We want to test the null hypothesis $\mathbf{P} = \Lambda(v^{\mathbf{P}}(\boldsymbol{\theta}))$, that consists of $2(A-1)|\mathcal{X}|$ constrains on $(\boldsymbol{\theta}, \mathbf{P})$. We can use a standard LM test. Under the null hypothesis, the LM statistic is asymptotically distributed as a chi-square with $2(A-1)|\mathcal{X}|$ degrees of freedom.

imposing the assumption of equilibrium beliefs can lead to significant bias in estimates of preferences and beliefs. We would like to understand the magnitude of this bias in the context of a simple application. Third, our identification results rest on an exclusion restriction: to identify one players payoffs and beliefs, there must be a special variable which does not enter his own payoffs (directly), but does enter the other player's payoffs. Clearly, the exclusion itself is important, but in finite sample one must also be concerned with how much the excluded variable affects the other player's payoffs, or how much it "shifts" the other player's behavior. If the special variable does not shift the other players behavior very much, we will have problems identifying our objects of interest. It is interesting from a practical standpoint to know just how significant a role this plays in identification.

Together the results of the Monte Carlo experiments help to illustrate the trade-off a researcher faces when deciding whether or not to impose the assumption of equilibrium beliefs, and how this trade off depends on the underlying DGP. On the one hand, by imposing the assumption of equilibrium beliefs the researcher is able to rely on the identification power afforded by the equilibrium restrictions, which results in more precise estimates, though he must be concerned with the possibility of biased results (if beliefs are not actually in equilibrium in the underlying DGP). On the other hand by not imposing the assumption of equilibrium beliefs the researcher does not have to be concerned with the bias caused by making an incorrect assumption about players' beliefs in the underlying DGP, but must be concerned with decreased precision in the estimates.

The model we consider in our experiments is a particular case of the *dynamic game of market entry and exit* in Example 1. We consider a game with two players. The per period profit functions of the two players are given by:

$$
\begin{aligned}
\pi_{1mt}(1, Y_{2mt}, \mathbf{X}_{mt}) &= (1 - Y_{3mt}) \, \theta_1^M + Y_{2mt} \, \theta_1^D - \theta_{01}^{FC} - (1 - Y_{1mt-1}) \, \theta_1^{EC} \\
\pi_{2mt}(1, Y_{1mt}, \mathbf{X}_{mt}) &= (1 - Y_{1mt}) \, \theta_2^M + Y_{1mt} \, \theta_2^D - \theta_{02}^{FC} - \theta_{12}^{FC} \, Z_{2m} - (1 - Y_{2mt-1}) \, \theta_2^{EC}
\end{aligned}
\tag{32}
$$

We normalize the profits to not being active to be zero for both players: $\pi_{1mt}(0, Y_{2mt}, \mathbf{X}_{mt}) = \pi_{2mt}(0, Y_{1mt}, \mathbf{X}_{mt}) = 0$. The players' payoffs to being active are symmetric except for the variable Z_{2m} which enters player 2's payoffs but not player 1's. Z_{2m} is an exogenous time-invariant characteristic which affects the fixed cost of player 2, but does not have a (direct) effect on the payoff of player 1. We assume that Z_{2m} is observable to the researcher. The vector of state variables is given by $\mathbf{X}_{mt} = (Z_{2m}, Y_{1mt-1}, Y_{2mt-1})$.

The model parameters to be estimated are $\{\theta_1^M, \theta_1^D, \theta_{01}^{FC}, \theta_1^{EC}\}$. Given the payoff structure we have described here, only the payoffs and beliefs of player 1 are identified under our identification assumptions.[13] The variable Z_{2m} allows us to identify the payoffs and beliefs of player 1. As we discuss and illustrate with the Monte Carlo experiments below, the value of θ_{12}^{FC}, which determines the sensitivity of player 2's payoffs to the variable Z_{2m} plays an important role in our ability to

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[13]Specifically, there is no variable with at least three points of support (i.e., $A+1 = 3$) that enters player 1's payoffs directly and does not enter player 2's payoffs directly. In principle Y_{1mt-1} could play the role of the "special variable" for identifying player 2's payoffs and beliefs, but since it can only take two values it is always at an "extreme point." In this case player 1's beliefs about player 2's behavior would always be correct. Relatedly, notice that in this set up we actually have over-identification, as the variable Y_{imt-1} is excluded from player j's payoffs for $i = 1, 2$. While these restrictions could be exploited in principle we do not do so here.

identify our objects of interest.

As we are only concerned with identification and estimation of payoffs and beliefs of player 1, we focus our discussion on these for the remainder of the section. Note that given the payoff structure we have assumed, and more specifically, given that we do not include market size H_{mt} as a state variable, we cannot separately identify the market-invariant component of fixed cost, θ_{01}^{FC}, from the fixed component of the variable profit, θ_1^M, as, regardless of the value of the vector $\mathbf{X}_{mt} = (Z_m, Y_{1mt-1}, Y_{2mt-1})$, the two parameters enters player 1's payoff in the exact same way. So define the parameters $\alpha_1 \equiv \theta_1^M - \theta_{01}^{FC} - \theta_1^{EC}$, and $\delta_1 \equiv \theta_1^M - \theta_1^D$. We can re-write player 1's per period profit function as:

$$\pi_{1mt}(1, Y_{2mt}, \mathbf{X}_{mt}) = \alpha_1 - \delta_1 \ Y_{jmt} + Y_{imt-1} \ \theta_1^{EC} \tag{33}$$

The exogenous variable Z_{2m} is independently and identically distributed over markets, with a discrete uniform distribution with support $\{-2, -1, 0, 1, 2\}$. As we mentioned above, Z_{2m} is the key variable which allows us to identify the payoffs and beliefs of player 1. Essentially Z_{2m} plays the role of an instrument for identifying the payoffs and beliefs of player 1 in the sense that it satisfies the usual exclusion restriction. It affects player 1's payoffs only through its effect on the behavior of player 1. As such, when using finite samples in practice, one must be concerned with the strength of the instrument. In our set-up here, the strength of the instrument, for a given support and distribution of Z_{2m} is completely determined by the value of θ_{12}^{FC}. By considering different values of θ_{12}^{FC} in the DGP, we use the Monte Carlo experiments to illustrate how a "weak instrument" may affect inference on payoffs and beliefs in a finite sample.

Table 3 presents the features of the DGPs in our experiments. We consider different values of θ_{12}^{FC} in the experiments, but keep α_i, δ_i, θ_i^{EC}, β_i constant. In order to provide an economic interpretation for the magnitude of these parameters, the table includes also some ratios implied by the payoff parameters.

In each experiment we consider, the sample is comprised by $M = 2,000$ markets and $T = 5$ time periods. We approximate the finite sample distribution of the estimators using $10,000$ Monte Carlo replications. The initial conditions for $\{Y_{1mt-1}, Y_{1mt-1}\}$ at $t = 1$ are drawn uniformly at random, as is the time invariant market specific variable Z_{2m}.

We implement four experiments: 1A, 1B, 2A, and 2B. The difference between experiments 1* and 2* is in the value of the parameter θ_{12}^{FC} associated with the power of the instrument Z_2: the value of this parameter is -0.5 in the 1A and 1B experiments, and -1.0 for the 2A and 2B experiments. The difference between experiments *A and *B is in the bias of players' beliefs. In experiments 1A and 2A beliefs are in equilibrium, while in experiments 1B and 2B players have biased beliefs. We now describe how players beliefs are determined in experiments 1B and 2B.

For for every player $i \in \{1, 2\}$ beliefs are $B_{imt}(\mathbf{X}_{mt}) = \lambda_{im} \ P_{jmt}(\mathbf{X}_{mt})$, where $\lambda_{im} \in [0, 1]$ is a parameter that captures player i's bias in beliefs in market m. Then, given λ_{1m} and λ_{2m}, the choice probabilities $P_{1mt}(\mathbf{X}_{mt})$ and $P_{2mt}(\mathbf{X}_{mt})$ solve a fixed point problem that we could describe as a *biased beliefs Markov Perfect Equilibrium* such that $P_{imt}(y|\mathbf{X}_{mt}) = \Lambda\left(y; \widetilde{\boldsymbol{v}}_{it}^{\mathbf{B}}(\mathbf{X}_{mt})\right)$ and $\mathbf{B}_{im} = \lambda_{im} \ \mathbf{P}_{jm}$. In Experiments 1A and 2A we fix $\lambda_{1m} = \lambda_{2m} = 1$ for every market m, such that

beliefs are unbiased. In Experiments 1B and 2B we fix the following values for the bias parameters λ_{im}:

$$\lambda_{im} = \begin{cases} 1 & \text{if} \quad Z_{2m} \in \{-2, 2\} \\ \\ 0.5 & \text{if} \quad Z_{2m} \in \{-1, 0, 1\} \end{cases} \tag{34}$$

That is, if the exogenous characteristic Z_{2m} is at an 'extreme' value, i.e., $Z_m \in \{-2, 2\}$, then there is not any strategic uncertainty or bias beliefs: players' beliefs are in equilibrium. However, if Z_{2m} lies in the interior of the support set, then beliefs are biased. More specifically, when beliefs are biased, both players are over-optimistic such that they underestimate (by 50%) the probability of the opponent will be active in the market. Note that given our choice of distribution of Z_{2m}, beliefs are (on average) out of equilibrium at 60% of the sample observations.[14]

Tables 4 and 5 summarize the results of our experiments. The tables report mean values and standard deviations from the Monte Carlo distribution of the estimators. As mentioned above, our interest in these experiments is threefold: to evaluate the loss of precision of our estimates when we relax the assumption of unbiased beliefs; to study the consequences of imposing the assumption of equilibrium beliefs when this assumption does not hold in the DGP; and to examine the role of the exclusion restriction in the precision of our estimates.

(a) Benchmark. Columns (1) and (2), in tables 4 and 5, present biases and standard errors of estimates when beliefs are unbiased in the DGP and we impose this restriction in the estimation. Relative to the true values of the parameters, biases and standard errors are always smaller than 5% and 10%, respectively. Therefore, we have a chosen a benchmark with quite precise estimates of payoffs and beliefs.

(b) Loss of precision when relaxing the assumption of unbiased beliefs. The main purpose of experiments 1A and 2A in each table is to evaluate the loss in identification power in finite sample when we do not impose the restrictions of equilibrium beliefs. With that purpose, we compare biases and standard errors in columns (3) and (4), where we relax the assumption of unbiased beliefs, with those in columns (1) and (2). The message in tables 2 and 3 are similar with respect to the identification power of equilibrium beliefs. Biases and standard errors increase substantially when we do not enforce the assumption of equilibrium beliefs. For instance, for the payoff parameter δ_1 that measures the competition effect, in Table 4, the bias goes from 3.35% to 4.83%, and the standard error from 7.83% to 12.54%. The loss of precision in the estimation of the entry cost parameter is more substantial: the bias goes from 0.42% to 15.20% and the standard error from 13.30% to 22.35%. Nevertheless, when we do not impose equilibrium restrictions, the estimates are still quite informative about the true value of the parameters, and 95% confidence interval are meaningful.

(c) Consequences of imposing the assumption of equilibrium beliefs when it is not true. In experiments 1B and 2B the DGP is such that beliefs are not in equilibrium. These experiments should help us to understand the bias induced by imposing the assumption of equilibrium beliefs

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[14] As a way of checking for possible coding errors in our program, we have also run all our experiments with $M = 100,000$ market observations instead of $2,000$. For the estimators of the correctly specified model, these experiments provide values of bias and standard deviations which are zero up to the fourth decimal place.

incorrectly, as well as the trade-off between bias and variance in the estimation without imposing equilibrium restrictions. Thus these experiments are informative for a researcher in that the help to clearly establish the costs and benefits of imposing the assumption of equilibrium beliefs when beliefs may not in equilibrium. The bias induced by the incorrect assumption of equilibrium beliefs is substantial. For instance, in Table 4 column (5), the bias in the estimate of δ_1 is almost 25% of the true value, and for the entry cost parameter the bias is more than 62% of the true value. These biases reduce drastically when we relax the assumption of equilibrium beliefs: in column (7) of Table 4, we find that the biases of the parameters δ_1 and θ_1^{EC} become 2.1% and 2.4% of their true values, respectively. In terms of the non parametric estimates of beliefs, the message is similar: enforcing equilibrium beliefs significantly improves precision at the cost of significantly biased estimates. When we incorrectly impose equilibrium beliefs, the bias in the estimate of beliefs is roughly 100% of the true value, while the bias is usually much less than 10% if we do not impose the assumption. Though the precision of the estimates decreases significantly when we do not impose the assumption, the combination of bias and variance show that, in this example, there are very significant gains in the estimates of payoffs and beliefs when we allow for biased beliefs.

(d) Quality of the instrument. Comparing across tables 4 and 5, we see a general improvement in both the bias and precision of estimates in all cases when we go from the case of $\theta_{12}^{FC} = 0.5$ to $\theta_{12}^{FC} = 1.0$. This is sensible, particularly with respect to the improved precision. In the case of $\theta_{12}^{FC} = 1.0$ we simply have a stronger instrument for player 2's entry decision, because player 2's decision is more sensitive to the value of Z_{2m}. As such, our ability to identify the parameters of player 1's payoff function should improve. This is also illustrated in figure 1 below, where we plot the mean squared error of our estimates of the parameters of player 1's payoff function from Monte Carlo experiments with 10 different values of θ_{12}^{FC}, from $\theta_{12}^{FC} = -0.1$ to $\theta_{12}^{FC} = -1.0$. Clearly, as the instrument becomes stronger (θ_{12}^{FC} increases in absolute value), the mean squared error of the parameter estimates decreases.

6 Empirical Application

We illustrate our model and methods with an application of a dynamic game of store location. Recently there has been significant interest in the estimation of game theoretic models of market entry and store location by retail firms. Most studies have assumed static games: see Mazzeo (2002), Seim (2006), Jia (2008), Nishida (2008), and Zhu and Singh (2009), among others. Holmes (2010) estimates a single-agent dynamic model of store location by Wal-Mart. Beresteanu and Ellickson (2005), Walrath (2008), and Suzuki (2010) propose and estimate dynamic games of store location.

We study store location of McDonalds (MD) and Burger King (BK) using data for the United Kingdom during the period 1991-1995. The dataset was collected by Otto Toivanen and Michael Waterson, who use it in their paper Toivanen and Waterson (2005).[15] We divide the UK into local markets (districts) and study these companies' decision of how many stores, if any, to operate in

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[15]We want to thank Otto Toivanen and Michael Waterson for generously sharing their data with us.

each local market. The profits of a store in a market depends on local demand and cost conditions and on the degree of competition from other firms' stores and from stores of the same chain. There are sunk costs associated with opening a new store, and therefore this decision has implications for future profits. Firms are forward-looking and maximize the value of expected and discounted profits. Each firm has uncertainty about future demand and cost conditions in local markets. Firms also have uncertainty about the current and future behavior of the competitor. In this context, the standard assumption is that firms have rational expectations about other firms' strategies, and that these strategies constitute a Markov Perfect Equilibrium. Here we relax this assumption. The main question that we want to analyze in this empirical application is whether the beliefs of each of these companies about the store location strategy of the competitor are consistent with the actual behavior of the competitor.

6.1 Data and descriptive evidence

Our working sample is a five year panel that tracks 422 *local authority districts* (local markets), including the information on the stock and flow of MD and BK stores into each district. It also contains socioeconomic variables at the district level such as population, density, age distribution, average rent, income per capita, local retail taxes, and distance to the UK headquarters of each of the firms. The *local authority district* is the smallest unit of local government in the UK, and generally consists of a city or a town sometimes with a surrounding rural area. There are almost 500 *local authority districts* in Great Britain. Our working sample of 422 districts does not include those that belong to Greater London.[16] The median district in our sample has an area of 300 square kilometers and a population of 95,000 people.[17] Table 6 presents descriptive statistics for socioeconomic and geographic characteristics of our sample of local authority districts.

Table 7 presents descriptive statistics on the evolution of the number of stores for the two firms.[18] In 1990, MD had more than three times the number of stores of BK, and it was active in more than twice the number of local markets than BK. Conditional on being active in a local market, MD had also significantly more stores per market than BK. These differences between MD and BK have not declined significantly over the period 1991-1995. While BK has entered in more new local markets than MD (69 new markets for BK and 48 new markets for MD), MD has opened more stores (143 new stores for BK and 166 new stores for MD).

Table 8 presents the annual transition probabilities of market structure in local markets as described by the number of stores of the two firms. According to this transition matrix, opening a new store is an irreversible decision, i.e., no store closings are observed during this sample period. In Britain during our sample period, the fast food hamburger industry was still young and expanding,

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[16] The reason we exclude the districts in Greater London from our sample is that they do not satisfy the standard criteria of isolated geographic markets.

[17] As a definition of geographic market for the fast food retail industry, the district is perhaps a bit wide. However, an advantage of using district as definition of local market is that most of the markets in our sample are geographically isolated. Most districts contain a single urban area. And, in contrast to North America where many fast food restaurants are in transit locations, in UK these restaurants are mainly located in the centers of urban areas.

[18] Toivanen and Waterson present a detailed discussion of why the retail chain fast food hamburger industry in the UK during this period can be assumed as a duopoly of BK and MD.

as shown by the large proportion of observations/local markets without stores (41.6%). Although there is significant persistence in every state, the less persistent market structures are those where BK is the leader. For instance, if the state is "$BK = 1$ & $MD = 0$", there is a 20% probability that the next year MD opens at least one store in the market. Similarly, when the state is "$BK = 2$ & $MD = 1$", the chances that MD opens one more store the next year are 31%.

Table 9 presents estimates of reduced form Probit models for the decision to open a new store. We obtain separate estimates for MD and BK. Our main interest is in the estimation of the effect of the previous year's number of stores (own stores and competitor's stores) on the probability of opening new stores. We include as control variables population, GDP per capita, population density, proportion of population 5-14, proportion population 15-29, average rent, and proportion of claimants of unemployment benefits. To control for unobserved local market heterogeneity we also present two fixed effects estimations, one with county fixed effects and the other with local district fixed effects. We only report estimates of the marginal effects associated with the dummy variables that represent previous year number of stores. The main empirical result from table 9 is that, regardless of the set of control variables that we use, the own number of stores has a strong negative effect on the probability of opening a new store but the effect of the competitor's number of stores is either negligible or even positive. This finding is very robust to different specifications of the reduced form model and it is analogous to the result from the reduced form specifications in Toivanen and Waterson (2005). Controlling for unobserved heterogeneity using fixed effects reveals that the estimate of the marginal effect of the number of own stores without fixed effects suffers from significant upward bias. However, the estimated marginal effect of the number of competitor's stores barely changes. The estimates show also a certain asymmetry between the two firms: the absence of response to the competitor's number of stores is more clear for BK than for MD. In particular, when BK has three stores in the market there is a significant reduction in MD's probability of opening a new store. That negative effect does not appear in the reduced form probit for BK.

This empirical evidence cannot be explained by a standard static model of store location by firms that sell substitute products. Here we explore three, non-mutually exclusive, explanations: (a) spillover effects; (b) forward looking behavior (dynamic game); and (c) biased beliefs about the behavior of the competitor.

(a) Spillover effects. The competitor's presence may have a positive spillover effect on the profit of a firm. There are several possible sources of this spillover effect. For example one firm may infer from another's decision to open a store in a particular market that market conditions are favorable (informational spillover effects). Alternatively, one firm may benefit from another firm's entry through cost reductions, or from product expansion through advertising. As such, we allow for the possibility of spillover effects in our specification of demand, but since we do not have price and quantity data at the level of local markets, we do not try to identify the source of the spillover effect. While the natural interpretation of the spillover effect in the context of our model is a product expansion due to an advertising effect of retail stores, this should be interpreted as a 'reduced form' specification of different possible spillover effects.

(b) Forward looking behavior. Opening a store is a partly irreversible decision that involves significant sunk costs. Therefore, it is reasonable to assume that firms are forward looking when they make this decision. Moreover, dynamic strategic effects may help explain the apparent absence of competitive effects when we study behavior in the context of a static model of entry. Suppose that firms anticipate, with some uncertainty, the total number of hamburger stores that a local market can sustain in the long-run given the size and the socioeconomic characteristics of the market. For simplicity, suppose that this number of "available slots" does not depend on the ownership of the stores because the products sold by the two firms are very close substitutes. In this context, firms play a game where they 'race' to fill as many 'slots' as possible with their own stores. Diseconomies of scale and scope may generate a negative effect of the own number of stores on the decision of opening new stores. However, in this model, during most of the period of expansion the number of slots of the competitor has zero effect on the decision of opening a new store. Only when the market is filled or close to being filled do the competitor's stores have a significant effect on entry decisions.

(c) Biased beliefs. As mentioned in the Introduction, competition in actual oligopoly industries is often characterized by strategic uncertainty. Firms face significant uncertainty about the strategies of their competitors. In the context of our application, it may be the case that MD's or/and BK's beliefs overestimate the negative effect of the competitor's stores on the competitor's entry decisions. For instance, if MD has one store in a local market, BK may believe that the probability that MD opens a second store is close to zero. These over-optimistic beliefs about the competitor's behavior may generate an apparent lack of response of BK's entry decisions to the number of MD's stores.

6.2 Model

Consider two retail chains competing in a local market. Each firm sells a differentiated product using its stores. Let $K_{imt} \in \{0, 1, ..., |\mathcal{K}|\}$ be the state variable that represents the number of stores of firm i in market m at period $t - 1$. And let $Y_{imt} \in \{0, 1, ..., A - 1\}$ be the number of *new* stores that firm i opens in the market during period t.[19] Following the empirical evidence during our sample period, we assume that opening a store is an irreversible decision. Also, for almost all the observations in the data we have that $Y_{imt} \in \{0, 1\}$, and therefore we consider a binary choice model for Y_{imt}, i.e., $A = 2$.

The total number of stores of firm i in market m at period t is $K_{imt} + Y_{imt}$. Firm i is active in the market at period t if $(K_{imt} + Y_{imt})$ is strictly positive. Every period, the two firms know the 'stocks' of stores in the market, K_{imt} and K_{jmt}, and simultaneously choose the new (additional) number of stores, Y_{imt} and Y_{jmt}. Firm i's total profit function is equal to variable profits minus entry costs and minus fixed operating costs: $\Pi_{imt} = VP_{imt} - EC_{imt} - FC_{imt}$.

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[19] We abstract from store location within a local market and assume that every store of the same firm has the same demand.

The specification of the variable profit function is:

$$VP_{imt} = (\mathbf{W}_m \ \gamma) \ (Y_{imt} + K_{imt}) \ \left[\theta_{0i}^{VP} + \theta_{can,i}^{VP}(K_{imt} + Y_{imt}) + \theta_{com,i}^{VP}(K_{jmt} + Y_{jmt})\right] \tag{35}$$

$\mathbf{W}_m$ is a vector of exogenous market characteristics such as population, population density, percentage of population in age group 15-29, GDP per capita, and unemployment rate. γ is a vector of parameters where the coefficient associated to the Population variable in $\mathbf{W}_{mt}$ is normalized to one. Therefore, the index $\mathbf{W}_m\gamma$ is measured in number of people and we interpret it as "market size". According to this specification, the term $\theta_{0i}^{VP} + \theta_{can,i}^{VP} \ (K_{imt} + Y_{imt}) + \theta_{com,i}^{VP} \ (K_{jmt} + Y_{jmt})$ represents variable profits per-capita and per-store. $\theta_{0i}^{VP} + \theta_{can,i}^{VP}$ is the variable profit (per capita) when firm i has a single store in the market. The term $\theta_{can,i}^{VP}(K_{imt} + Y_{imt})$ captures cannibalization effects between stores of the same chain as well as possible economies of scale and scope in variable costs. Term $\theta_{com,i}^{VP}(K_{jmt} + Y_{jmt})$ captures the effect of competition from the other chain.

Entry cost have the following form:

$$EC_{imt} = 1\{Y_{imt} > 0\} \ \left[\theta_{0i}^{EC} + \theta_{K,i}^{EC} \ 1\{K_{imt} > 0\} + \theta_{Z,i}^{EC} \ Z_{imt} + \varepsilon_{it}\right] \tag{36}$$

$1\{.\}$ is the indicator function, and θ_{0i}^{EC}, $\theta_{K,i}^{EC}$, and $\theta_{Z,i}^{EC}$ are parameters. θ_{0i}^{EC} is an entry cost that is paid the first time that the firm opens a store in the local market. $\theta_{0i}^{EC} + \theta_{K,i}^{EC}$ is the cost of opening a new store when the firm already has stores in the market. If there are economies of scope in the operation of multiple stores in a market, we expect the parameter $\theta_{K,i}^{EC}$ to be negative such that the entry cost of the first store is greater than the entry cost of additional stores. Z_{imt} represents the geographic distance between market m and the closest market where firm i has stores at period $t-1$ (i.e., Z_{imt} is zero if $K_{imt} > 0$). The term $\theta_{Z,i}^{EC} \ Z_{imt}$ tries to capture economies of density as in Holmes (2010). The random variable ε_{it} is a private information shock in the cost of opening a new store, and it is i.i.d. normally distributed.

The specification of fixed costs is:

$$FC_{imt} = 1\{(K_{imt} + Y_{imt}) > 0\} \ \left[\theta_{0i}^{FC} + \theta_{lin,i}^{FC}(K_{imt} + Y_{imt}) + \theta_{qua,i}^{FC}(K_{imt} + Y_{imt})^2\right] \tag{37}$$

θ_{0i}^{FC} is a lump-sum cost associated with having any positive number of stores in the market. The term $\theta_{lin,i}^{FC} \ (K_{imt} + Y_{imt}) + \theta_{qua,i}^{FC} \ (K_{imt} + Y_{imt})^2$ takes into account that operating costs may increase (or decline) with the number of stores in a quadratic form.

Given this specification, the vector of state variables S_{imt} is (K_{imt}, Z_{imt}). A firm's variable profit depends on his own and his opponents current number of stores in the market, and also on his own stock of stores at previous period, K_{imt}, and on the distance from market m to the closest store of the chain at year $t-1$. These two variables, K_{imt} and Z_{imt}, affect the entry cost of firm i in market m. However, the competitors' number of stores in the previous year, and the distance from market m to the closest store of the competitor in the previous year, do not directly affect the current profit of the firm. This satisfies the exclusion restriction in assumption ID-3. Of course a firm's beliefs about the probability distribution of the opponents' choice, Y_{jmt}, depend on $S_{jmt} = (K_{jmt}, Z_{jmt})$.

The maximum value of K_{imt} in the sample is 13, but K_{imt} is less than or equal to three for 99% of the observations in the sample. We assume that the set of possible values of K_{imt} is $\{0, 1, 2, 3\}$, where $K_{imt} = 3$ represents a number of stores greater or equal than three. When $K_{imt} = 3$, we impose the restriction that firm i does not open additional stores in this market: $P_{imt}(1|\mathbf{X}_{mt}$ with $K_{imt} = 3) = 0$. The variable Z_{imt}, that represents the distance to the closest chain store, is discretized into 8 cells of 30 miles intervals: $Z_{imt} = 1$ represents a distance of less than 30 miles, $Z_{imt} = 2$ for a distance of between 30 and 60 miles, ..., $Z_{imt} = 7$ for a distance of between 180 and 210 miles, and $Z_{imt} = 8$ for a distance greater than 210 miles. Market characteristics in the vector $\mathbf{W}_m$ have very little time variability in our sample and we treat them as time invariant state variables in order to reduce the dimensionality of the state space. Therefore, the set $\mathcal{S}$ is equal to $\{0, 1, 2, 3\} \times \{1, 2, ..., 8\}$ and it has 32 grid points, and the whole state space $\mathcal{X}$ is equal to $\mathcal{S} \times \mathcal{S}$ and it has $1,024$ points.

Assumption ID-4, which restricts beliefs over a subset of the state space, takes the following form in this application. We assume that the two firms have unbiased beliefs about the entry behavior of the opponent in markets which are relatively close to the opponents network, i.e., for small values of the distance Z_{jmt}. However, beliefs may be biased for markets that are farther away to the opponent's network. More formally, we assume that:

$$B_{imt}(y_j|\mathbf{X}_{mt}) = P_{jmt}(y_j|\mathbf{X}_{mt}) \qquad \text{if } Z_{jmt} \leq Z^* \tag{38}$$

We have estimated the model for different values of Z^*. The main intuition behind this assumption is that markets that are far away from a firm's network are unexplored markets for which there is more strategic uncertainty.

Our assumption on players' beliefs implies that the degree of bias in firms' beliefs declines over time with the geographic expansion of these retail chains. Eventually, when the retail chains have sufficiently expanded geographically, the beliefs of firms become unbiased for every market and state. More formally, with probability one, there is a period in the future, say t^* with $t^* < \infty$, such that for any $t \geq t^*$, any market m, and any firm j, we have that $Z_{jmt} \leq Z^*$. It is straightforward to check if condition $Z_{jmt} \leq Z^*$ is satisfied for every market and firm in the data after some year in the sample, such that we can say that the sample period includes year t^*. For our choices of Z^*, this condition is almost, but not exactly, satisfied in the last year of our sample, 1995. However, we need to impose this restriction to apply our backwards induction identification and estimation procedure in a dynamic game with infinite horizon. Therefore, in all our estimations, we assume that beliefs are in equilibrium at every year t greater than or equal to the last year in our sample, 1995.

6.3 Estimation of the structural model

Table 10 presents estimates of the dynamic game under three different assumptions on beliefs. Columns (1) and (2) present estimates under the assumption that beliefs are unbiased for every value of the state variables. In columns (3) and (4), we impose the restriction of unbiased beliefs only when the distance to the competitor's network is shorter than 60 miles, i.e., $Z^* = 2$. In

columns (5) and (6), beliefs are unbiased when that distance is shorter than 30 miles , i.e., $Z^* = 1$. For each of these three scenarios, the proportion of observations at year 1995 for which we impose the restriction of unbiased beliefs is 100%, 38%, and 29%, respectively.

(a) Estimation with unbiased beliefs. The estimation shows substantial differences between estimated parameters in the variable profit function of the two firms. The parameter θ_{can}^{VP} is negative and significant for BK but positive and also statistically significant for MD. Cannibalization effects dominate in the case of BK. In contrast, economies of scope in variable profits seem important for MD. The estimates of the parameter that captures the competitive effect, θ_{com}^{VP}, are smaller in magnitude than the estimates of θ_{can}^{VP}, but they are statistically significant. According to these estimates the competitive effect of MD's market presence on BK's profits is smaller than the reverse effect.

The estimates of fixed cost parameters illustrates a similarity across firms in the structure of fixed costs of operation. The fixed operating cost increases linearly, not quadratically, with the number of stores, and the lump-sum component of the cost is relatively small. However, there a substantial economic differences between the firms in the magnitude of these costs. The fixed cost that BK pays per additional store is almost twice the fixed cost MD pays.

Entry costs are particularly important in this setting because they play a key role in the identification of the dynamic game, through the exclusion restrictions. The estimates of these costs are very significant, both statistically and economically. Entry costs depend significantly on the number of installed stores of the firm, K, and on the distance to the firm's network, Z. The signs of these effects, negative for θ_K^{EC} and positive for θ_Z^{EC}, are consistent with the existence of economies of scope and density between the stores of the same chain. McDonalds has smaller entry costs, and a larger absolute value of the parameter θ_K^{EC}, which indicates that there are stronger economies of scope in the network of McDonalds stores.

In summary, the estimated model with unbiased beliefs shows significant differences in the variable profits and entry costs of the firms. Cannibalization is stronger between BK stores, while MD exhibits substantial economies of scope both in variables profits and entry costs. Competition effects seem relatively weak but statistically significant.

(b) Tests of unbiased beliefs. Our test of unbiased beliefs clearly rejects the null hypothesis for BK, with a p-value of 0.00029, though we cannot reject the hypothesis of unbiased beliefs for MD.[20]

(c) Estimation with biased beliefs. As expected, (bootstrap) standard errors increase significantly when we estimate the model allowing for biased beliefs. Nevertheless, these standard errors are not large and the estimation provides informative and meaningful results. Comparing these parameter estimates with those in the model with equilibrium restrictions, the most important changes are in the parameters of variable profits of BK. In particular, the estimate of the parameter that measures the competitive effect of MD on BK is now more than twice the initial estimate with equilibrium beliefs. In contrast to the result with unbiased beliefs, we find that the competitive effect of MD on BK is stronger that the effect of BK on MD. This result is consistent with the findings in our Monte Carlo experiments: imposing the restriction of unbiased beliefs when it is

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[20] To implement this test we use a vector $\widehat{\delta}_i = \{\widehat{\delta}_i(S_i) : S_i \in \mathcal{S}\}$ of $|\mathcal{S}| = 32$ statistics.

incorrect introduces a "measurement error" in beliefs which in turn generates an attenuation bias in the estimate of the parameter associated with the strategic interactions. For the identification of this structural parameter the sample variation in beliefs plays an important role.

Interestingly, BK's estimated profit function has a lower level when we allow for biased beliefs than when we enforce unbiased beliefs: variable profits are lower, and fixed costs and entry costs are larger. This is fully consistent with our finding that the bias in BK's beliefs are mostly in the direction of underestimating the true probability that MD will enter in unexplored markets. If we impose the assumption of unbiased beliefs, BK's profit must be relatively high in order to rationalize entry into markets where MD is also likely to enter or to expand its number of stores. Once we take into account the over-optimistic beliefs of BK about the behavior of MD, revealed preference shows that BK profits are not as high as before. In fact, in the estimates that allow for biased beliefs we find that the differences in the profit function of MD and BK are even larger.

7   Conclusion

This paper studies a class of dynamic games of incomplete information where players' beliefs about the other players' actions may not be in equilibrium. We present new results on identification, estimation, and inference of structural parameters and beliefs in this class of games when the researcher does not have data on elicited beliefs, or these data are limited to players' beliefs at only some values of the state variables. Specifically, we derive sufficient conditions under which payoffs and beliefs are point identified. These conditions then lead naturally to a sequential estimator of payoffs and beliefs. We also present a procedure for testing the null hypothesis that beliefs are in equilibrium. We illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by McDonalds and Burger King. They key conditions for the identification of beliefs and payoffs in our application are the following. The first condition is an exclusion restriction in a firm's profit function that establishes that the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit through the existence of sunk entry costs and economies of density in these costs. The second condition restricts firms' beliefs to be unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores. However, beliefs are unrestricted, and potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs for Burger King. We find that Burger King underestimated the probability of entry of McDonalds in markets that were relatively far away from McDonalds' network of stores.

APPENDIX

[A.1] Integrated Value Function and Continuation Values

Our proofs of Propositions 1 and 2 apply the concepts of *integrated value function* and *continuation value function* as well as recursive formulas to calculate these functions. The *integrated value function* is defined as $\bar{V}_{it}^{\mathbf{B}}(\mathbf{X}_t) \equiv \int V_{it}^{\mathbf{B}}(\mathbf{X}_t, \varepsilon_{it}) \, dG_{it}(\varepsilon_{it})$ (see Rust, 1994). Applying this definition to the Bellman equation, we obtained the *integrated Bellman equation:*

$$\bar{V}_{it}^{\mathbf{B}}(\mathbf{X}_t) = \int \max_{y_i \in \mathcal{Y}} \left\{ v_{it}^{\mathbf{B}}(y_i, \mathbf{X}_t) + \varepsilon_{it}(y_i) \right\} \, dG_{it}(\varepsilon_{it})$$

$$= \int \max_{y_i \in \mathcal{Y}} \left\{ \pi_{it}^{\mathbf{B}}(y_i, \mathbf{X}_t) + \beta \sum_{\mathbf{X}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1}) \, f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|y_i, \mathbf{X}_t) + \varepsilon_{it}(y_i) \right\} dG_{it}(\varepsilon_{it})$$

(A.1.1)

If $\{\varepsilon_{it}(0), \varepsilon_{it}(1), ..., \varepsilon_{it}(A-1)\}$ are i.i.d. extreme value type 1, the integrated Bellman equation has the following closed-form expression:

$$\bar{V}_{it}^{\mathbf{B}}(\mathbf{X}_t) = \ln \left(\sum_{y_i \in \mathcal{Y}} \exp \left\{ v_{it}^{\mathbf{B}}(y_i, \mathbf{X}_t) \right\} \right)$$

(A.1.2)

If we knew payoffs and beliefs, we could use this formula to obtain the integrated value function by backwards induction, starting at the last period T where $\bar{V}_{iT}^{\mathbf{B}}(\mathbf{X}) = \ln \left(\sum_{y_i \in \mathcal{Y}} \exp \left\{ \pi_{it}^{\mathbf{B}}(y_i, \mathbf{X}_t) \right\} \right)$.

The *continuation value function* provides the expected and discounted value of *future* payoffs given future beliefs of player i and current choices of all the players. It is defined as:

$$c_{it}(\mathbf{Y}_t, \mathbf{X}_t) \equiv \beta \sum_{\mathbf{X}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1}) \, f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$$

(A.1.3)

Note that continuation values c_{it} depend on beliefs at periods $t+1$ and later, but not on beliefs at period t. By definition, the relationship between the conditional choice value function $v_{it}^{\mathbf{B}}$ and the continuation value function c_{it} is the following:

$$v_{it}^{\mathbf{B}}(y_i, \mathbf{X}_t) = \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} [\pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{X}_t) + c_{it}(y_i, \mathbf{y}_{-i}, \mathbf{X}_t)] \, B_{it}(\mathbf{y}_{-i}|\mathbf{X}_t)$$

(A.1.4)

[A.2] Proof of Proposition 1

[Part (i): Identification of payoffs] The restrictions of the model that come from best response behavior of player i can be represented using the following equation. For any $(y_i, \mathbf{X}) \in \mathcal{Y} \times \mathcal{X}$,

$$q_{it}(y_i, \mathbf{X}) = \mathbf{B}_{it}(\mathbf{X})' \, [\boldsymbol{\pi}_{it}(y_i, \mathbf{X}) + \widetilde{\mathbf{c}}_{it}(y_i, \mathbf{X})]$$

(A.2.1)

where $\mathbf{B}_{it}(\mathbf{X})$, $\boldsymbol{\pi}_{it}(y_i, \mathbf{X})$, and $\widetilde{\mathbf{c}}_{it}(y_i, \mathbf{X})$ are vectors with dimension $A^{N-1} \times 1$ containing beliefs, payoffs, and continuation values, respectively, for every possible value of $\mathbf{y}_{-i}$ in the set $\mathcal{Y}^{N-1}$. Let $\mathcal{S}_{-i}^{(R)}$ be the set $[\mathcal{S}^{(R)}]^{N-1}$. By assumption ID-4, for any $\mathbf{X}$ such that $\mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}$ we have that $B_{it}(\mathbf{y}_{-i}|\mathbf{X}) = P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ and $P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ is known to the researcher. Consider the system of equations formed by equation (A.2.1) at a fixed value of $(y_i, S_i, \mathbf{W})$ and for every value of $\mathbf{S}_{-i}$ in

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$\mathcal{S}_{-i}^{(R)}$. This is a system of R^{N-1} equations, and we can represent this system in vector form using the following expression:

$$\widetilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i) \;\; = \mathbf{P}_{-it}^{(R)}(S_i)\, \boldsymbol{\pi}_{it}(y_i, S_i) \tag{A.2.2}$$

where: $\boldsymbol{\pi}_{it}(y_i, S_i)$ is the $A^{N-1} \times 1$ vector $\{\pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{S}_{-i}) : \mathbf{y_{-i}} \in \mathcal{Y}^{\mathbf{N-1}}\}$; $\mathbf{P}_{-it}^{(R)}(S_i)$ is the $R^{N-1} \times A^{N-1}$ matrix $\{P_{-it}(\mathbf{y}_{-i}|S_i, \mathbf{S}_{-i}) : \mathbf{y_{-i}} \in \mathcal{Y}^{\mathbf{N-1}}, \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$; and $\widetilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i)$ is the $R^{N-1} \times 1$ vector with elements $\{\widetilde{q}_{it}(y_i, S_i, \mathbf{S}_{-i}, \mathbf{W}) : \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$ where

$$\widetilde{q}_{it}(y_i, \mathbf{X}) \equiv q_{it}(y_i, \mathbf{X}) - \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} P_{-it}(\mathbf{y}_{-i}|\mathbf{X})\, c_{it}(y_i, \mathbf{y}_{-i}, \mathbf{X})$$

Under condition (i) in Proposition 1, matrix $\mathbf{P}_{-it}^{(R)}(S_i)'\mathbf{P}_{-it}^{(R)}(S_i)$ is non-singular and therefore we can solve for vector $\boldsymbol{\pi}_{it}(y_i, S_i)$ in the previous system of equations:

$$\boldsymbol{\pi}_{it}(y_i, S_i) = \left[\mathbf{P}_{-it}^{(R)}(S_i)'\, \mathbf{P}_{-it}^{(R)}(S_i)\right]^{-1} \mathbf{P}_{-it}^{(R)}(S_i)'\, \widetilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i) \tag{(A.2.3)}$$

This expression shows that, given continuation values at period t, the vector of payoffs $\boldsymbol{\pi}_{it}(y_i, S_i)$ is identified, i.e., part (i) of Proposition 1.

[Part (ii): Identification of beliefs] Now, we show the identification of the beliefs function for states outside the subset $\mathcal{S}_{-i}^{(R)}$. Again, we start with the system equations implied by the best response restrictions, but now we take into account that the vector $\boldsymbol{\pi}_{it}(y_i, S_i)$ is identified and then look at the identification of beliefs at states $\mathbf{X}$ with $\mathbf{S}_{-i}$ outside the subset $\mathcal{S}_{-i}^{(R)}$. We stack equation (A.2.1) for every value of $y_i \in \mathcal{Y} - \{0\}$ to obtain a system of equations. Note that $\mathbf{B}_{it}(\mathbf{X})$ is a vector of A^{N-1} probabilities, one element for each value of $\mathbf{y}_{-i}$ in $\mathcal{Y}^{N-1}$. The probabilities in this vector should sum to one, and therefore, $\mathbf{B}_{it}(\mathbf{X})$ satisfies the restriction $\mathbf{1}'\mathbf{B}_{it}(\mathbf{X}) = 1$, where $\mathbf{1}$ is a vector of ones. Therefore, we have the following system of A equations:

$$\mathbf{q}_{it}(\mathbf{X}) \;\; = \widetilde{\mathbf{V}}_{it}(\mathbf{X})\, \mathbf{B}_{it}(\mathbf{X}) \tag{A.2.4}$$

$\mathbf{q}_{it}(\mathbf{X})$ is an $A \times 1$ vector with elements $\{q_{it}(1, \mathbf{X}), ..., q_{it}(A-1, \mathbf{X})\}$ at rows 1 to $A-1$, and a 1 at the last row. And $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is an $A \times A^{N-1}$ matrix with elements:

$$\pi_{it}(y_i, \mathbf{y}_{-i}, S_i, \mathbf{W}) + [c_{it}(y_i, \mathbf{y}_{-i}, \mathbf{X}) - c_{it}(0, \mathbf{y}_{-i}, \mathbf{X})] \tag{A.2.5}$$

and the last row of the matrix is a row of ones. When $N = 2$, matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is an $A \times A$ matrix, and condition (ii) in Proposition 1 implies that this matrix is non-singular. Therefore, if continuation values at period t are known to the researcher, we can identify beliefs in the vector $\mathbf{B}_{it}(\mathbf{X})$ as:

$$\mathbf{B}_{it}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X}) \tag{A.2.6}$$

Now, we prove that condition (ii) implies that matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is non-singular. Our proof of part (i) implies that:

$$\widetilde{\mathbf{V}}_{it}(\mathbf{X}) = \mathbf{Q}_{it}^{(R)}(S_i)\, \mathbf{P}_{-it}^{(R)}(S_i)\, \left[\mathbf{P}_{-it}^{(R)}(S_i)'\, \mathbf{P}_{-it}^{(R)}(S_i)\right]^{-1} \tag{A.2.7}$$

and $\mathbf{Q}_{it}^{(R)}(S_i)$ is the $A \times R$ matrix with $\mathbf{q}_{it}^{(R)}(y_i, S_i)'$ at the first $A-1$ rows, and ones at the last row. By Assumption ID-4, $\mathbf{P}_{-it}^{(R)}(S_i)$ is full column rank, and then a sufficient condition for $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ to be non-singular matrix is that $\mathbf{Q}_{it}^{(R)}(S_i)$ has rank A, which is a condition in part (ii) of Proposition 1.

[**Corollary: Full identification**] Given parts (i) and (ii) of Proposition 1, it is straightforward to show, using backwards induction, the identification of payoffs and beliefs at every period t. At the last period T, continuation values are zero, and therefore $\boldsymbol{\pi}_{iT}$ and $\mathbf{B}_{iT}$ are identified as:

$$\boldsymbol{\pi}_{iT}(y_i, S_i) = \left[\mathbf{P}_{-iT}^{(R)}(S_i)'\ \mathbf{P}_{-iT}^{(R)}(S_i)\right]^{-1} \mathbf{P}_{-iT}^{(R)}(S_i)'\ \mathbf{q}_{iT}^{(R)}(y_i, S_i) \tag{A.2.8}$$

and

$$\mathbf{B}_{iT}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{iT}(\mathbf{X})\right]^{-1} \mathbf{q}_{iT}(\mathbf{X}) \tag{A.2.9}$$

For any period $t < T$, given payoffs, beliefs, and continuation values at period $t+1$, we can construct continuation values at period t. First, we obtain conditional choice value functions at period $t+1$:

$$v_{it+1}^{\mathbf{B}}(y_i, \mathbf{X}) = \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} [\pi_{it+1}(y_i, \mathbf{y}_{-i}, \mathbf{X}) + c_{it+1}(y_i, \mathbf{y}_{-i}, \mathbf{X})]\ B_{it+1}(\mathbf{y}_{-i}|\mathbf{X}) \tag{A.2.10}$$

Second, we obtain the integrated value function at period $t+1$:

$$\bar{V}_{it+1}^{\mathbf{B}}(\mathbf{X}) = \ln\left(\sum_{y_i \in \mathcal{Y}} \exp\left\{v_{it+1}^{\mathbf{B}}(y_i, \mathbf{X})\right\}\right) \tag{A.2.11}$$

And finally, we calculate the continuation values at period t:

$$c_{it}(\mathbf{Y}_t, \mathbf{X}_t) = \beta \sum_{\mathbf{X}_{t+1} \in \mathcal{X}} \bar{V}_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1})\ f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t) \tag{A.2.12}$$

Given these continuation values, we apply the formulas in ((A.2.3)) and (A.2.6) to obtain payoffs and beliefs at t. By using backwards induction we identify beliefs and payoff functions at every period t. ∎

[A.3] Proof of Proposition 2

The proof has two parts. First, we show that given CCPs of player i only, it is possible to identify a function that depends on beliefs of players but not on payoffs. Second, under the assumption of equilibrium beliefs, the identified function of beliefs can be also identified using only CCPs of player j. Therefore, we have identified the same object using two different sources of data. If the hypothesis of equilibrium beliefs is correct, the two approaches should give us the same result, but if beliefs are biased the two approaches provide different results. This can be used to construct a test statistic.

There are $N = 2$ players, i and j, the vector of state variables $\mathbf{X}$ as $(S_i, S_j, \mathbf{W})$, and players' actions are y_i and y_j. Under the condition in Proposition 2 that the transition of the state variables has the form $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{W}_t)$, we have that continuation values $c_{it}(\mathbf{Y}_t, \mathbf{X}_t)$ do not depend on $\mathbf{S}_t$. Therefore, the restrictions of the model can be written as:

$$q_{it}(y_i, \mathbf{X}) = \mathbf{B}_{it}(\mathbf{X})'\ \widetilde{\mathbf{v}}_{it}(y_i, S_i, \mathbf{W}) \tag{A.3.1}$$

where $\mathbf{B}_{it}(\mathbf{X})$ is the $A \times 1$ vector defined above, and $\widetilde{\mathbf{v}}_{it}(y_i, S_i, \mathbf{W})$ is the $A \times 1$ vector with elements $\{\pi_{it}(y_i, y_j, S_i, \mathbf{W}) + \widetilde{c}_{it}(y_i, y_j, \mathbf{W}) : y_j \in \mathcal{Y}\}$. For notational simplicity and without loss of generality, we omit $\mathbf{W}$ for the rest of this proof.

Let s_j^0 be an arbitrary value of in the set $\mathcal{S}$. And let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be two different subsets included in the set $\mathcal{S} - \{s_j^0\}$ such that they satisfy two conditions: (1) each of these sets has $A-1$ elements; and (2) $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have at least one element that is different. Since $|\mathcal{S}| \geq A+1$, it is

always possible to construct two subsets that satisfy these conditions. Given one of these subsets, say $\mathcal{S}^{(a)}$, we can construct the following system of $A - 1$ equations:

$$\Delta \mathbf{q}_{it}^{(a)}(y_i, S_i) \quad = \Delta \mathbf{B}_{it}^{(a)}(S_i) \; \widetilde{\widetilde{\mathbf{v}}}_{it}(y_i, S_i) \tag{A.3.2}$$

where: $\Delta \mathbf{q}_{it}^{(a)}(y_i, S_i)$ is an $(A - 1) \times 1$ vector with elements $\{q_{it}(y_i, S_i, S_j) - q_{it}(y_i, S_i, s_j^0) :$ for $S_j \in \mathcal{S}^{(a)}\}$; $\Delta \mathbf{B}_{it}^{(a)}(S_i)$ is a $(A - 1) \times (A - 1)$ matrix with elements $\{B_{it}(y_j, S_i, S_j) - B_{it}(y_j, S_i, s_j^0) :$ for $y_j \in \mathcal{Y} - \{0\}$ and $S_j \in \mathcal{S}^{(a)}\}$; and $\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i, S_i)$ is a $(A - 1) \times 1$ vector with elements $\{\pi_{it}(y_i, y_j, S_i) + \widetilde{c}_{it}(y_i, y_j) - \pi_{it}(y_i, 0, S_i) - \widetilde{c}_{it}(y_i, 0) : y_j \in \mathcal{Y}\}$. Using the other subset, $\mathcal{S}^{(b)}$, we can construct a similar system of $A - 1$ equations. Given that matrices $\Delta \mathbf{B}_{it}^{(a)}(S_i)$ and $\Delta \mathbf{B}_{it}^{(b)}(S_i)$ are non-singular, we can use these systems to obtain to different solutions for $\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i, S_i)$:

$$
\begin{aligned}
\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i, S_i) \quad &= \quad \left[\Delta \mathbf{B}_{it}^{(a)}(S_i)\right]^{-1} \Delta \mathbf{q}_{it}^{(a)}(y_i, S_i) \\[2mm]
&= \quad \left[\Delta \mathbf{B}_{it}^{(b)}(S_i)\right]^{-1} \Delta \mathbf{q}_{it}^{(b)}(y_i, S_i)
\end{aligned}
\tag{A.3.3}
$$

For given S_i, we have these two solutions of $\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i, S_i)$ for every value of y_i in the set $\mathcal{Y} - \{0\}$. Putting these $A - 1$ solutions in matrix form, we have:

$$\left[\Delta \mathbf{B}_{it}^{(a)}(S_i)\right]^{-1} \Delta \mathbf{Q}_{it}^{(a)}(S_i) \quad = \quad \left[\Delta \mathbf{B}_{it}^{(b)}(S_i)\right]^{-1} \Delta \mathbf{Q}_{it}^{(b)}(S_i) \tag{A.3.4}$$

where $\Delta \mathbf{Q}_{it}^{(a)}(S_i)$ and $\Delta \mathbf{Q}_{it}^{(b)}(S_i)$ are $(A - 1) \times (A - 1)$ matrices with columns $\Delta \mathbf{q}_{it}^{(a)}(y_i, S_i)$ and $\Delta \mathbf{q}_{it}^{(b)}(y_i, S_i)$, respectively. Given that $\Delta \mathbf{Q}_{it}^{(a)}(S_i)$ is an invertible matrix, we can rearrange the previous system in the following way:

$$\Delta \mathbf{B}_{it}^{(a)}(S_i) \left[\Delta \mathbf{B}_{it}^{(b)}(S_i)\right]^{-1} \quad = \quad \Delta \mathbf{Q}_{it}^{(a)}(S_i) \left[\Delta \mathbf{Q}_{it}^{(b)}(S_i)\right]^{-1} \tag{A.3.5}$$

This expression shows that we can identify the $(A-1) \times (A-1)$ matrix $\Delta \mathbf{B}_{it}^{(a)}(S_i) \left[\Delta \mathbf{B}_{it}^{(b)}(S_i)\right]^{-1}$ that depends only on beliefs, using only the CCPs of player i. That is, we can identify $(A-1) \times (A-1)$ objects or functions of beliefs.

Under the assumption of unbiased beliefs, we can use the CCPs of the other player, j, to identify matrix $\Delta \mathbf{B}_{it}^{(a)}(S_i) \left[\Delta \mathbf{B}_{it}^{(b)}(S_i)\right]^{-1}$:

$$\Delta \mathbf{B}_{it}^{(a)}(S_i) \left[\Delta \mathbf{B}_{it}^{(b)}(S_i)\right]^{-1} \quad = \quad \Delta \mathbf{P}_{jt}^{(a)}(S_i) \left[\Delta \mathbf{P}_{jt}^{(b)}(S_i)\right]^{-1} \tag{A.3.6}$$

where $\Delta \mathbf{P}_{jt}^{(a)}(S_i)$ is $(A - 1) \times (A - 1)$ matrix with elements $\{P_{jt}(y_j, S_i, S_j) - P_{jt}(y_j, S_i, s_j^0) :$ for $y_j \in \mathcal{Y} - \{0\}$ and $S_j \in \mathcal{S}^{(a)}\}$, and $\Delta \mathbf{P}_{jt}^{(b)}(S_i)$ has a similar definition. Therefore, under the assumption of unbiased beliefs by player i the CCPs of player i and player j should satisfy the following $(A - 1)^2$ restrictions:

$$\Delta \mathbf{Q}_{it}^{(a)}(S_i) \left[\Delta \mathbf{Q}_{it}^{(b)}(S_i)\right]^{-1} - \Delta \mathbf{P}_{jt}^{(a)}(S_i) \left[\Delta \mathbf{P}_{jt}^{(b)}(S_i)\right]^{-1} \quad = \quad \mathbf{0} \tag{A.3.7}$$

These restrictions are testable. ∎

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Table 1

Sequence of Beliefs $B_{jt}^{(t_0)}$

Period when beliefs are formed (t_0)	Period of the opponents' behavior (t)					
	$t=1$	$t=2$	$t=3$	...	$t=T-1$	$t=T$
$t_0=1$	$B_{i1}^{(1)}$	$B_{i2}^{(1)}$	$B_{i3}^{(1)}$	...	$B_{i,T-1}^{(1)}$	$B_{iT}^{(1)}$
$t_0=2$	-	$B_{i2}^{(2)}$	$B_{i3}^{(2)}$	...	$B_{i,T-1}^{(2)}$	$B_{iT}^{(2)}$
$t_0=3$	-	-	$B_{i3}^{(3)}$	...	$B_{i,T-1}^{(3)}$	$B_{iT}^{(3)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t_0=T-1$	-	-	-	...	$B_{i,T-1}^{(T-1)}$	$B_{iT}^{(T-1)}$
$t_0=T$	-	-	-	...	-	$B_{iT}^{(T)}$

Table 2
Restrictions, Unknowns, and Identification Order Condition

Number of parameters & restrictions per (i,t) and per possible free value of (Y_i, X)

Conditions on the Model	**(1)** # Restrictions	**(2)** # Parameters in payoffs	**(3)** # Parameters in beliefs	**(1) - (2) - (3)** # Over (+) Under (-) Restrictions
Unrestricted beliefs and payoffs	1	A^{N-1}	$\dfrac{A^{N-1}-1}{A-1}$	$1 - A^{N-1} - \dfrac{A^{N-1}-1}{A-1}$ No Identification if $N \geq 2$
Equil. beliefs, unrestricted payoff	$1+\dfrac{A^{N-1}-1}{A-1}$	A^{N-1}	$\dfrac{A^{N-1}-1}{A-1}$	$1 - A^{N-1}$ No Identification if $N \geq 2$
Equil. beliefs, exclusion restriction	$1+\dfrac{A^{N-1}-1}{A-1}$	$\dfrac{A^{N-1}}{|S|^{N-1}}$	$\dfrac{A^{N-1}-1}{A-1}$	$1 - \dfrac{A^{N-1}}{|S|^{N-1}}$ Identification for any N if $|S| \geq A$
Unrestricted beliefs, exclusion restriction	1	$\dfrac{A^{N-1}}{|S|^{N-1}}$	$\dfrac{A^{N-1}-1}{A-1}$	$1 - \dfrac{A^{N-1}}{|S|^{N-1}} - \dfrac{A^{N-1}-1}{A-1}$ No Identification if $N \geq 2$
Partially rest. beliefs exclusion restriction	$1+\dfrac{R^{N-1}}{|S|^{N-1}}\left(\dfrac{A^{N-1}-1}{A-1}\right)$	$\dfrac{A^{N-1}}{|S|^{N-1}}$	$\dfrac{A^{N-1}-1}{A-1}$	$1 - \dfrac{A^{N-1}}{|S|^{N-1}}$ $+\left(\dfrac{R^{N-1}}{|S|^{N-1}} - 1\right)\dfrac{A^{N-1}-1}{A-1}$

Table 3
Summary of DGPs in the Monte Carlo Experiments

For all the experiments: $\alpha_i = 2.4;$ $\delta_i = 3.0;$ $\theta_i^{EC} = 0.5;$ $\beta_i = 0.95;$
$$Z_{2m} \sim Uniform \ \{-2, -1, 0, +1, +2\}$$
$$M = 2,000; \ T = 5; \ MC \ replications = 10,000$$

Experiment 1A:	$\theta_{12}^{FC} = -0.5;$	Unbiased beliefs
Experiment 1B:	$\theta_{12}^{FC} = -0.5;$	Biased beliefs
Experiment 2A:	$\theta_{12}^{FC} = -1.0;$	Unbiased beliefs
Experiment 2B:	$\theta_{12}^{FC} = -1.0;$	Biased beliefs

Some Ratios Implied by these Parameter Values

Entry cost over profit of average monopolist: $\theta_i^{EC}/(\theta_i^M - \theta_{0i}^{FC})$ 17.1%

Profit reduction from monopoly to duopoly: $(\theta_i^M - \theta_i^D)/(\theta_i^M - \theta_{0i}^{FC})$ 103.4%

Profit reduction for player 2 as monopolist from $Z_2 = -2$ to $Z_i = 2$:
$\theta_{12}^{FC}(2 - (-2)) \ / \ (\theta_2^M - \theta_{02}^{FC} - (-2)\theta_{12}^{FC})$

                                                with $\theta_{12}^{FC} = 0.5$ 51.3%
                                                 with $\theta_{12}^{FC} = 1.0$ 81.6%

Table 4
Monte Carlo Experiments 1A and 1B

	Experiment 1A DGP: $\theta_{12}^{FC}=-0.5$; Unbiased beliefs					Experiment 1B DGP: $\theta_{12}^{FC}=-0.5$; Biased beliefs			
	(1)	**(2)**	**(3)**	**(4)**		**(5)**	**(6)**	**(7)**	**(8)**
	Estimation with equilibrium restrictions		Estimation no equilibrium restrictions			Estimation with equilibrium restrictions		Estimation no equilibrium restrictions	
Parameter (True value)	Bias (%)	Std (%)	Bias (%)	Std (%)		Bias (%)	Std (%)	Bias (%)	Std (%)
Payoffs					Payoffs				
α_i (2.4)	-0.0992	0.2208	0.1412	0.3702	α_i (2.4)	0.0157	0.2999	-0.0657	0.3312
	(4.13)	(9.20)	(5.88)	(15.42)		(0.65)	(12.50)	(2.74)	(13.80)
δ_i (3.0)	-0.1004	0.2349	0.1448	0.3763	δ_i (3.0)	0.7481	0.3250	0.0617	0.3488
	(3.35)	(7.83)	(4.83)	(12.54)		(24.94)	(10.83)	(2.06)	(11.63)
θ_i^{EC} (0.5)	-0.0021	0.0665	-0.0760	0.1118	θ_i^{EC} (0.5)	-0.3114	0.0798	-0.0120	0.1478
	(0.42)	(13.30)	(15.20)	(22.35)		(62.28)	(15.96)	(2.41)	(29.55)
Beliefs: $t=1$; $Z_2=0$: $B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6993)	-0.0005	0.0460	0.0109	0.1563	$B_{1t}(0,0)$ (0.4145)	-0.4144	0.0378	0.0108	0.2401
	(0.07)	(6.57)	(1.57)	(22.36)		(99.99)	(9.12)	(2.61)	(57.92)
$B_{1t}(0,1)$ (0.8390)	0.0004	0.0369	0.0204	0.1259	$B_{1t}(0,1)$ (0.4454)	-0.4449	0.0313	0.0178	0.2464
	(0.05)	(4.40)	(2.43)	(15.01)		(99.88)	(7.03)	(3.99)	(55.32)
$B_{1t}(1,0)$ (0.6009)	-0.0001	0.0488	0.0169	0.1850	$B_{1t}(1,0)$ (0.4078)	-0.4076	0.0388	0.0083	0.2466
	(0.02)	(8.13)	(2.82)	(30.79)		(99.99)	(9.52)	(2.05)	(60.48)
$B_{1t}(1,1)$ (0.7603)	0.0002	0.0423	0.0159	0.1544	$B_{1t}(1,1)$ (0.4405)	0.4403	0.0323	0.0136	0.2500
	(0.02)	(5.56)	(2.09)	(20.31)		(99.99)	(7.33)	(3.08)	(56.75)
Beliefs: $t=5$; $Z_2=0$; $B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6269)	-0.0018	0.0966	0.0099	0.2268	$B_{1t}(0,0)$ (0.3876)	-0.3765	0.1672	-0.0025	0.3006
	(0.29)	(15.41)	(1.58)	(36.17)		(97.14)	(43.14)	(0.63)	(77.56)
$B_{1t}(0,1)$ (0.8034)	-0.0002	0.0448	0.0330	0.1937	$B_{1t}(0,1)$ (0.4272)	-0.4271	0.0530	-0.0108	0.2822
	(0.02)	(5.58)	(4.11)	(24.11)		(99.99)	(12.42)	(2.52)	(66.07)
$B_{1t}(1,0)$ (0.4975)	0.0014	0.0568	-0.0266	0.1855	$B_{1t}(1,0)$ (0.3768)	-0.3768	0.0655	-0.0066	0.2047
	(0.28)	(11.41)	(5.35)	(37.30)		(99.99)	(17.39)	(1.75)	(54.32)
$B_{1t}(1,1)$ (0.6939)	0.0003	0.0315	0.0012	0.0756	$B_{1t}(1,1)$ (0.4190)	0.4187	0.0211	0.0073	0.1139
	(0.05)	(4.54)	(0.17)	(10.90)		(99.94)	(5.04)	(1.75)	(27.19)

Table 5
Monte Carlo Experiments 2A and 2B

	Experiment 2A DGP: $\theta_{12}^{FC}=-1.0$; Unbiased beliefs					Experiment 2B DGP: $\theta_{12}^{FC}=-1.0$; Biased beliefs			
	(1)	(2)	(3)	(4)		(5)	(6)	(7)	(8)
	Estimation with equilibrium restrictions		Estimation no equilibrium restrictions			Estimation with equilibrium restrictions		Estimation no equilibrium restrictions	
Parameter	Bias	Std	Bias	Std	Parameter	Bias	Std	Bias	Std
(True value)	(%)	(%)	(%)	(%)		(%)	(%)	(%)	(%)
Payoffs					Payoffs				
α_i (2.4)	-0.0726	0.1832	0.2060	0.3045	α_i (2.4)	-0.3332	0.2666	-0.0081	0.2829
	(3.03)	(7.63)	(8.58)	(12.69)		(13.88)	(11.11)	(0.34)	(11.79)
δ_i (3.0)	-0.0793	0.1852	0.1802	0.3048	δ_i (3.0)	0.2979	0.2746	0.1543	0.3071
	(2.64)	(6.17)	(6.01)	(10.16)		(9.93)	(9.15)	(5.14)	(10.24)
θ_i^{EC} (0.5)	-0.0042	0.0629	-0.0861	0.1229	θ_i^{EC} (0.5)	-0.3277	0.0778	0.0134	0.1482
	(0.84)	(12.57)	(17.21)	(24.59)		(65.55)	(15.56)	(2.68)	(29.63)
Beliefs: $t=1$; $Z_2=0$: $B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6993)	-0.0005	0.0460	0.0257	0.1683	$B_{1t}(0,0)$ (0.4145)	-0.4144	0.0378	0.0283	0.2449
	(0.07)	(6.57)	(3.68)	(24.07)		(99.99)	(9.12)	(6.83)	(59.08)
$B_{1t}(0,1)$ (0.8390)	0.0004	0.0369	0.0308	0.1318	$B_{1t}(0,1)$ (0.4454)	-0.4449	0.0313	0.0389	0.2457
	(0.05)	(4.40)	(3.67)	(15.70)		(99.88)	(7.03)	(8.74)	(55.15)
$B_{1t}(1,0)$ (0.6009)	-0.0001	0.0488	0.0341	0.1982	$B_{1t}(1,0)$ (0.4078)	-0.4076	0.0388	0.0270	0.2506
	(0.02)	(8.13)	(5.68)	(32.99)		(99.99)	(9.52)	(6.61)	(61.45)
$B_{1t}(1,1)$ (0.7603)	0.0002	0.0423	0.0286	0.1638	$B_{1t}(1,1)$ (0.4405)	0.4403	0.0323	0.0359	0.2525
	(0.02)	(5.56)	(3.76)	(21.55)		(99.99)	(7.33)	(8.16)	(57.33)
Beliefs: $t=5$; $Z_2=0$; $B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6269)	-0.0018	0.0966	0.0505	0.2644	$B_{1t}(0,0)$ (0.3876)	-0.3765	0.1672	0.0104	0.3080
	(0.29)	(15.41)	(8.06)	(42.17)		(97.14)	(43.14)	(2.68)	(79.47)
$B_{1t}(0,1)$ (0.8034)	-0.0002	0.0448	0.0066	0.1776	$B_{1t}(0,1)$ (0.4272)	-0.4271	0.0530	-0.1357	0.2419
	(0.00)	(5.58)	(0.83)	(22.11)		(99.99)	(12.42)	(31.77)	(56.63)
$B_{1t}(1,0)$ (0.4975)	0.0014	0.0568	-0.0387	0.2538	$B_{1t}(1,0)$ (0.3768)	-0.3768	0.0655	-0.0498	0.2789
	(0.28)	(11.41)	(7.77)	(51.03)		(99.99)	(17.39)	(13.22)	(74.02)
$B_{1t}(1,1)$ (0.6939)	0.0003	0.0315	0.0060	0.0969	$B_{1t}(1,1)$ (0.4190)	0.4187	0.0211	0.0110	0.1469
	(0.05)	(4.54)	(0.86)	(13.97)		(99.94)	(5.04)	(2.63)	(35.06)

Table 6
Descriptive Statistics on Local Markets (Year 1991)

422 local authority districts (excluding Greater London districts)

Variable	Median	Std. Dev.	Pctile 5%	Pctile 95%
Area (thousand square km)	0.30	0.73	0.03	1.67
Population (thousands)	94.85	93.04	37.10	280.50
Children: Age 5-14 (%)	12.43	1.00	10.74	14.07
Young: 15-29 (%)	21.24	2.46	17.80	25.17
Pensioners: 65-74 (%)	9.01	1.50	6.89	11.82
GDP per capita (thousand £)	92.00	12.14	74.40	112.70
Claimants of UB / Population ratio (%)	2.75	1.27	1.24	5.11
Avg. Weekly Rent per dwelling (£)	25.31	10.61	19.11	35.07
Council tax (thousand £)	0.24	0.05	0.11	0.31
Number of BK stores	0.00	0.62	0.00	1.00
Number of MD stores	1.00	1.16	0.00	3.00

Table 7
Evolution of the Number of Stores

422 local authority districts (excluding Greater London districts)

	Burger King					
	1990	**1991**	**1992**	**1993**	**1994**	**1995**
#Markets with Stores	71	98	104	118	131	150
Change in #Markets with Stores	-	17	6	14	13	19
# of Stores	79	115	128	153	181	222
Change in # of Stores	-	36	13	25	28	41
Mean #Stores per Market (Conditional on #Stores>0)	1.11	1.17	1.23	1.30	1.38	1.48
	McDonalds					
	1990	**1991**	**1992**	**1993**	**1994**	**1995**
#Markets with Stores	206	213	220	237	248	254
Change in #Markets with Stores		7	7	17	11	6
# of Stores	281	316	344	382	421	447
Change in # of Stores		35	28	38	39	26
Mean #Stores per Market (Conditional on #Stores>0)	1.36	1.49	1.56	1.61	1.70	1.76

Table 8
Reduced Form Probits for the Decision to Open a Store

| | Estimated Marginal Effects[1] ($\Delta P(x)$ when dummy goes from 0 to 1) | | | | | |
| | Burger King | | | McDonalds | | |
Explanatory Variable	No FE	County FE	District FE	No FE	County FE	District FE
Own number of stores at t-1						
Dummy: Own #stores = 1	-0.021**	-0.036**	-0.885**	-0.035**	-0.045**	-0.550**
	(0.005)	(0.007)	(0.063)	(0.010)	(0.012)	(0.056)
Dummy: Own #stores = 2	-0.023**	-0.030**	-0.210*	-0.047**	-0.060*	-0.757**
	(0.004)	(0.005)	(0.085)	(0.006)	(0.008)	(0.041)
Dummy: Own #stores $\geq$ 3	-0.019**	-0.027**	-0.056	-0.043**	-0.053**	-0.816**
	(0.005)	(0.005)	(0.036)	(0.006)	(0.008)	(0.038)
Competitor's number of stores at t-1						
Dummy: Comp.'s #stores = 1	0.032**	0.037*	-0.025	0.020	0.032*	0.052**
	(0.011)	(0.014)	(0.055)	(0.013)	(0.018)	(0.073)
Dummy: Comp.'s #stores = 2	0.045*	0.052*	-0.017	0.041	0.076	-0.007**
	(0.023)	(0.029)	(0.031)	(0.029)	(0.046)	(0.093)
Dummy: Comp.'s #stores $\geq$ 3	0.089*	0.101*	0.011	-0.041**	-0.050**	-0.104**
	(0.048)	(0.059)	(0.084)	(0.007)	(0.009)	(0.020)
Pred. Prob. Y=1 at mean X	0.024	0.027	0.014	0.045	0.054	0.085
Time dummies	YES	YES	YES	YES	YES	YES
Control variables[2]	YES	YES	YES	YES	YES	YES
County Fixed Effects	NO	YES	NO	NO	YES	NO
District Fixed Effects	NO	NO	YES	NO	NO	YES
Number of Observations[3]	2110	1715	535	2110	1855	640
Number of Local Districts[3]	422	343	107	422	371	128
log likelihood	-371.89	-340.26	-110.54	-467.46	-449.02	-198.50
Pseudo R-square	0.229	0.252	0.624	0.159	0.161	0.441

Note 1: Estimated Marginal Effects are evaluated at the mean value of the rest of the explanatory variables.
Note 2: Every estimation includes as control variables log of population, log of GDP per capita, log of population density, proportion population 5-14, proportion population 15-29, average rent, and proportion of claimants of unemployment benefits.
Note 3: Fixed effects estimations do not include districts for which the dependent variable does not have enough time variation.

Table 9
Transition Probability Matrix for Market Structure
Annual Transitions. Market structure: BK=x & MD=y, where x and y are number of stores

	% Market Structure at t+1								
	BK=0 MD=0	BK=0 MD=1	BK=0 MD≥ 2	BK=1 MD=0	BK=1 MD=1	BK=1 MD≥ 2	BK≥ 2 MD=0	BK≥ 2 MD=1	BK≥ 2 MD≥ 2
BK=0 &MD=0	**95.1**	3.6	0.2	1.0	-	-	-	0.1	-
BK=0 &MD=1	-	**87.2**	4.2	-	7.4	1.0	-	-	1.4
BK=0 &MD≥ 2	-	-	**82.7**	-	-	15.8	-	-	1.4
BK=1 &MD=0	-	-	-	**76.0**	18.0	2.0	4.0	-	-
BK=1 &MD=1	-	-	-	-	**87.1**	8.1	-	3.3	1.4
BK=1 &MD≥ 2	-	-	-	-	-	**86.5**	-	-	13.5
BK≥ 2 &MD=0	-	-	-	-	-	-	**84.6**	15.4	-
BK≥ 2 &MD=1	-	-	-	-	-	-	-	**69.0**	31.0
BK≥ 2 &MD≥ 2	-	-	-	-	-	-	-	-	**100.0**
Frequency	41.6	23.3	6.6	2.2	10.9	8.8	0.6	1.4	4.5

Table 10
Estimation of Dynamic Game for McDonalds and Burger King
**Models with Unbiased and Biased Beliefs**[1]

Data: 422 markets, 2 firms, 5 years = 4,220 observations

|	$\beta = 0.95$ (not estimated)					
	Unbiased Beliefs		Biased Beliefs: $Z^* = 2$		Biased Beliefs: $Z^* = 1$	
	Burger King	McDonalds	Burger King	McDonalds	Burger King	McDonalds
Variable Profits:						
θ_0^{VP}	0.5413	0.8632	0.4017	0.8271	0.4342	0.8582
	$(0.1265)^*$	$(0.2284)^*$	$(0.2515)^*$	$(0.4278)^*$	(0.2820)	(0.4375)
θ_{can}^{VP} cannibalization	-0.2246	0.0705	-0.2062	0.0646	-0.1926	0.0640
	$(0.0576)^*$	$(0.0304)^*$	$(0.1014)^*$	(0.0710)	$(0.1140)^*$	(0.0972)
θ_{com}^{VP} competition	-0.0541	-0.0876	-0.1133	-0.0856	-0.1381	-0.0887
	$(0.0226)^*$	(0.0272)	$(0.0540)^*$	(0.0570)	$(0.0689)^*$	(0.0622)
Fixed Costs:						
θ_0^{FC} fixed	0.0350	0.0374	0.0423	0.0307	0.0490	0.0339
	(0.0220)	(0.0265)	(0.0478)	(0.0489)	(0.0585)	(0.0658)
θ_{lin}^{FC} linear	0.0687	0.0377	0.0829	0.0467	0.0878	0.0473
	$(0.0259)^*$	$(0.0181)^*$	$(0.0526)^*$	(0.0291)	(0.0665)	(0.0344)
θ_{qua}^{FC} quadratic	-0.0057	0.0001	-0.0007	0.0002	-0.0004	0.0004
	(0.0061)	(0.0163)	(0.0186)	(0.0198)	(0.0253)	(0.0246)
Entry Cost:						
θ_0^{EC} fixed	0.2378	0.1887	0.2586	0.1739	0.2422	0.1764
	$(0.0709)^*$	$(0.0679)^*$	$(0.1282)^*$	$(0.0989)^*$	(0.1504)	(0.1031)
θ_K^{EC} (K)	-0.0609	-0.107	-0.0415	-0.1190	-0.0419	-0.1271
	(0.043)	$(0.0395)^*$	(0.096)	$(0.0628)^*$	$(0.109)^*$	$(0.0762)^*$
θ_Z^{EC} (Z)	0.0881	0.0952	0.1030	0.1180	0.0902	0.1212
	$(0.0368)^*$	$(0.0340)^*$	$(0.0541)^*$	$(0.0654)^*$	(0.0628)	$(0.0759)^*$
Log-Likelihood	-848.4		-840.4		-838.7	
Test of unbiased beliefs for BK: $\widehat{D}$ (d.o.f) (p-value)			66.841 (32) (0.00029)			
Test of unbiased beliefs for MD: $\widehat{D}$ (d.o.f) (p-value)			42.838 (32) (0.09549)			

Note 1: Bootstrap standard errors in parentheses.

FIGURE 1

Square−root MSE of payoff parameter ALPHA
vs. quality of the instrument
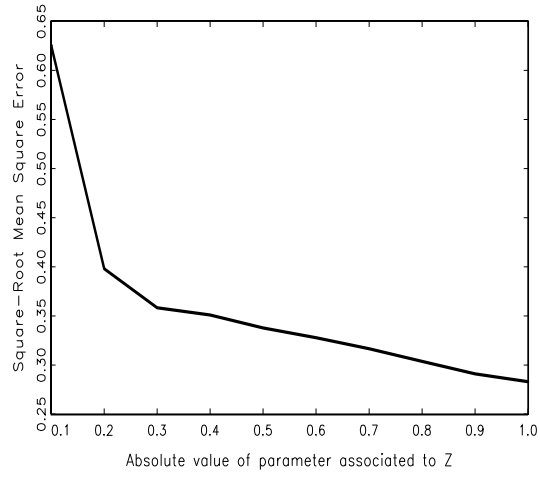
Square−root MSE of payoff parameter DELTA
vs. quality of the instrument
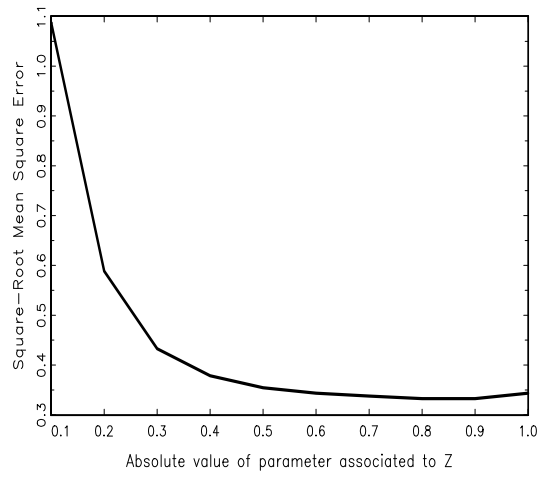
Square−root MSE of payoff parameter Entry Cost
vs. quality of the instrument

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