



Imposing local curvature in the QUAIDS[☆]

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ABSTRACT

In this paper, we build on Ryan and Wales (1998), Moschini (1999), and Serletis and Shahmoradi (2007) and impose curvature conditions locally on the quadratic Almost Ideal Demand System (AIDS) model of Banks et al. (1997), an extension of the simple AIDS model of Deaton and Muellbauer (1980) that can generate quadratic Engel curves [that is, rank-three demand systems, in the terminology of Lewbel (1991)]. In doing so, we exploit the Slutsky matrix of second order derivatives of the indirect utility function.

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1. Introduction

Locally flexible demand models such as the translog of Christensen et al. (1975), generalized Leontief of Diewert (1974), and almost ideal demand system (AIDS) of Deaton and Muellbauer (1980) have revolutionized microeconometrics, by providing access to all neoclassical microeconomic theory in econometric applications. However, as argued by Caves and Christensen (1980), Guilkey and Lovell (1980) and Barnett and Lee (1985); Barnett et al. (1985, 1987), among others, most popular locally flexible functional forms have very small regions of theoretical regularity, thereby giving up global integrability. This problem led to the development of locally-flexible functional forms, which have larger regularity regions. Cooper and McLaren (1996) classify such models as ‘effectively globally regular’. Examples of these functions include Barnett’s (1983; 1985) minflex Laurent models, based on the Laurent series expansion, the quadratic AIDS (QUAIDS) model of Banks et al. (1997), and the general exponential form (GEF) of Cooper and McLaren (1996). Barnett and Serletis (2008) provide a discussion of these models.

Although the usefulness of flexible functional forms depends on whether they satisfy the theoretical regularity conditions (of positivity, monotonicity, and curvature), in the empirical demand systems literature, there often has been a tendency to ignore theoretical regularity or not to report the results of full regularity checks. In fact, as Barnett (2002, p. 199) observed in his *Journal of Econometrics* Fellow’s opinion article, “without satisfaction of both curvature and monotonicity, the second-order conditions for optimizing behavior fail, and duality theory fails. The resulting first-order conditions, demand functions, and supply functions become invalid”. For this reason, in the recent literature, there is an attempt to treat theoretical regularity as a maintained hypothesis and built it into the models being estimated, very much like the homogeneity in prices and symmetry properties of neoclassical consumer theory.

In particular, Ryan and Wales (1998), drawing on related work by Lau (1978), Gallant and Golub (1984) and Diewert and Wales (1987), suggest a procedure for imposing local curvature conditions and apply their procedure to three locally flexible functional forms—the AIDS of Deaton and Muellbauer (1980), the normalized quadratic (NQ) of Diewert and Wales (1988), and the linear translog. Moreover, Moschini (1999) suggest a possible reparameterization of the basic translog of Christensen et al. (1975) to overcome some problems noted by Ryan and Wales (1998) and also impose curvature conditions locally in the basic translog. More recently, Serletis and Shahmoradi (2007) impose curvature conditions locally on the generalized Leontief model, introduced

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by Diewert (1974). In doing so, they exploit the Hessian matrix of second order derivatives of the reciprocal indirect utility function, unlike Ryan and Wales (1998) and Moschini (1999) who exploit the Slutsky matrix.

In this note, we build on Ryan and Wales (1998), Moschini (1999) and Serletis and Shahmoradi (2007) and impose curvature conditions locally on the QUAIDS model of Banks et al. (1997). In doing so, like Ryan and Wales (1998) and Moschini (1999), we exploit the Slutsky matrix of second order derivatives of the QUAIDS indirect utility function.

The rest of the paper is organized as follows. Section 2 briefly presents the QUAIDS model while Section 3 discusses our procedure for imposing local curvature conditions. Section 4 presents an empirical application and the final section provides a brief conclusion.

2. The QUAIDS

According to Banks et al. (1997), the QUAIDS has the following indirect utility function

$$\ln h(\mathbf{p}, y) = \left\{ \left[\frac{\ln y - \ln a(\mathbf{p})}{b(\mathbf{p})} \right]^{-1} + \lambda(\mathbf{p}) \right\}^{-1} \quad (1)$$

where y is total expenditure, \mathbf{p} is a vector of prices, $a(\mathbf{p})$ is a differentiable, homogeneous function of degree one in prices, and $\lambda(\mathbf{p})$ and $b(\mathbf{p})$ are differentiable, homogeneous functions of degree zero in prices. In fact, $\lambda(\mathbf{p})$ takes the form

$$\lambda(\mathbf{p}) = \sum_{i=1}^n \lambda_i \ln p_i, \quad \text{where } \sum_{i=1}^n \lambda_i = 0$$

and $i = 1, \dots, n$ denotes the number of goods.

The specifications of $a(\mathbf{p})$ and $b(\mathbf{p})$ in (1) are similar to those in the AIDS of Deaton and Muellbauer (1980); they are sufficiently flexible to represent any arbitrary set of first and second derivatives of the cost function, as follows

$$\ln a(\mathbf{p}) = \alpha_0 + \sum_{i=1}^n \alpha_i \ln p_i + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n \gamma_{ik} \ln p_i \ln p_k$$

$$b(\mathbf{p}) = \prod_{i=1}^n (p_i)^{\beta_i}.$$

Solving for $\ln y$, yields the QUAIDS cost function

$$\ln C \equiv \ln y = \ln a(\mathbf{p}) + \frac{b(\mathbf{p}) \ln h(\mathbf{p}, y)}{1 - \lambda(\mathbf{p}) \ln h(\mathbf{p}, y)}. \quad (2)$$

Applying Shephard's lemma to the cost function (2) or Roy's identity to the indirect utility function (1) yields the QUAIDS share equations

$$s_i = \alpha_i + \sum_{j=1}^n \gamma_{ij} \ln p_j + \beta_i \ln \left[\frac{y}{a(\mathbf{p})} \right] + \frac{\lambda_i}{b(\mathbf{p})} \left\{ \ln \left[\frac{y}{a(\mathbf{p})} \right] \right\}^2 \quad (3)$$

where s_i is the i th budget share and α , β , γ , and λ are parameters. By setting $\lambda_i = 0$ ($i = 1, \dots, n$), Eq. (3) reduces to the AIDS share equations.

Economic theory imposes several restrictions on the parameters of the model. In particular, Slutsky symmetry requires

$$\gamma_{ij} = \gamma_{ji}, \quad \text{for all } i, j.$$

Homogeneity of the Marshallian demand functions of degree zero in (\mathbf{p}, y) requires

$$\sum_{j=1}^n \gamma_{ij} = 0, \quad \text{for all } i.$$

Finally, the adding-up condition requires $\sum_{i=1}^n s_i = 1$ or, equivalently,

$$\sum_{i=1}^n \alpha_i = 1; \quad \sum_{i=1}^n \beta_i = 0;$$

$$\sum_{i=1}^n \lambda_i = 0; \quad \text{and} \quad \sum_{i=1}^n \gamma_{ij} = 0, \quad \text{for all } j.$$

3. Imposing local curvature

The Ryan and Wales (1998) procedure for imposing local curvature conditions applies to those locally flexible demand systems for which, at the point of approximation, the $n \times n$ Slutsky matrix \mathbf{S} can be written as

$$\mathbf{S} = \mathbf{B} + \mathbf{C} \quad (4)$$

where \mathbf{B} is an $n \times n$ symmetric matrix, containing the same number of independent elements as the Slutsky matrix, and \mathbf{C} is an $n \times n$ matrix whose elements are functions of the other parameters of the system. Curvature requires the Slutsky matrix to be negative semidefinite.

Ryan and Wales (1998) draw on related work by Lau (1978) and Diewert and Wales (1987) and impose curvature by replacing \mathbf{S} in (4) with $-\mathbf{K}\mathbf{K}'$, where \mathbf{K} is an $n \times n$ lower triangular matrix so that $-\mathbf{K}\mathbf{K}'$ is by construction a negative semidefinite matrix. Then solving explicitly for \mathbf{B} in terms of \mathbf{K} and \mathbf{C} yields

$$\mathbf{B} = -\mathbf{K}\mathbf{K}' - \mathbf{C}$$

meaning that the model can be reparameterized by estimating the parameters in \mathbf{K} and \mathbf{C} instead of the parameters in \mathbf{B} and \mathbf{C} . That is, we can replace the elements of \mathbf{B} in the estimating Eq. (3) by the elements of \mathbf{K} and the other parameters of the model, thus ensuring that \mathbf{S} is negative semidefinite at the point of approximation, which could be any data point.

Following this procedure, the ij th element of the QUAIDS Slutsky matrix at the reference point ($p^* = y^* = 1$) is

$$S_{ij} = \gamma_{ij} + (\alpha_0 \beta_i - \alpha_0^2 \lambda_i - \alpha_i) \delta_{ij} - \alpha_0 \beta_i \beta_j$$

$$+ \alpha_0^2 \beta_i \lambda_j + \alpha_0^2 \beta_j \lambda_i - 2\alpha_0^3 \lambda_i \lambda_j + \alpha_i \alpha_j$$

$$- \alpha_0 \alpha_i \beta_j - \alpha_0 \alpha_j \beta_i + \alpha_0^2 \alpha_i \lambda_j + \alpha_0^2 \alpha_j \lambda_i$$

$$+ \alpha_0^2 \beta_i \beta_j - \alpha_0^3 \beta_i \lambda_j - \alpha_0^3 \beta_j \lambda_i + \alpha_0^4 \lambda_i \lambda_j$$

where δ_{ij} is the Kronecker delta, which equals 1 when $i = j$ and 0 otherwise. By replacing \mathbf{S} by $-\mathbf{K}\mathbf{K}'$, the above can be written as

$$(-\mathbf{K}\mathbf{K}')_{ij} = \gamma_{ij} + (\alpha_0 \beta_i - \alpha_0^2 \lambda_i - \alpha_i) \delta_{ij} - \alpha_0 \beta_i \beta_j$$

$$+ \alpha_0^2 \beta_i \lambda_j + \alpha_0^2 \beta_j \lambda_i - 2\alpha_0^3 \lambda_i \lambda_j + \alpha_i \alpha_j$$

$$- \alpha_0 \alpha_i \beta_j - \alpha_0 \alpha_j \beta_i + \alpha_0^2 \alpha_i \lambda_j + \alpha_0^2 \alpha_j \lambda_i$$

$$+ \alpha_0^2 \beta_i \beta_j - \alpha_0^3 \beta_i \lambda_j - \alpha_0^3 \beta_j \lambda_i + \alpha_0^4 \lambda_i \lambda_j. \quad (5)$$

Solving for the γ_{ij} terms as a function of the $(\mathbf{K}\mathbf{K}')_{ij}$ terms we can get the restrictions that ensure the negative semidefiniteness of the Slutsky matrix, without destroying the flexibility of (1) as the number of free parameters remains the same.

As an example, for the case of three goods ($n = 3$), Eq. (5) implies the following four restrictions on the γ_{ij} terms

$$\gamma_{11} = -k_{11}^2 - \alpha_0 \beta_1 + \alpha_0^2 \lambda_1 + \alpha_1 + \alpha_0 \beta_1^2$$

$$- 2\alpha_0^2 \beta_1 \lambda_1 + 2\alpha_0^3 \lambda_1^2 - \alpha_1^2 + 2\alpha_0 \alpha_1 \beta_1 - 2\alpha_0^2 \alpha_1 \lambda_1$$

$$- \alpha_0^2 \beta_1^2 + 2\alpha_0^3 \beta_1 \lambda_1 - \alpha_0^4 \lambda_1^2$$

$$\gamma_{22} = -k_{22}^2 - k_{22}^2 - \alpha_0 \beta_2 + \alpha_0^2 \lambda_2 + \alpha_2 + \alpha_0 \beta_2^2 - 2\alpha_0^2 \beta_2 \lambda_2$$

$$+ 2\alpha_0^3 \lambda_2^2 - \alpha_2^2 + 2\alpha_0 \alpha_2 \beta_2 - 2\alpha_0^2 \alpha_2 \lambda_2 - \alpha_0^2 \beta_2^2$$

$$+ 2\alpha_0^3 \beta_2 \lambda_2 - \alpha_0^4 \lambda_2^2$$

Table 1
QUAIDS parameter estimates.

Parameter	Unrestricted	Local curvature imposed
a_0	10.549 (0.026)	10.603 (0.467)
a_1	0.161 (0.774)	0.160 (0.912)
a_2	0.481 (0.272)	0.488 (0.642)
β_{11}	-0.042 (0.588)	-0.137 (0.373)
β_{12}	-0.043 (0.477)	0.048 (0.666)
β_{22}	0.053 (0.296)	-0.049 (0.546)
b_1	-0.118 (0.571)	-0.099 (0.783)
b_2	-0.092 (0.659)	-0.072 (0.838)
λ_1	-0.022 (0.000)	-0.012 (0.000)
λ_2	-0.022 (0.000)	-0.012 (0.000)
Positivity violations	3	0
Monotonicity violations	3	0
Curvature violations	2	0
Log likelihood value	16,555.2	16,485.4

$$\gamma_{12} = -k_{11}k_{12} + \alpha_0\beta_1\beta_2 - \alpha_0^2\beta_1\lambda_2 - \alpha_0^2\beta_2\lambda_1 \\ + 2\alpha_0^3\lambda_1\lambda_2 - \alpha_1\alpha_2 + \alpha_0\alpha_1\beta_2 + \alpha_0\alpha_2\beta_1 - \alpha_0^2\alpha_1\lambda_2 \\ - \alpha_0^2\alpha_2\lambda_1 - \alpha_0^2\beta_1\beta_2 + \alpha_0^3\beta_1\lambda_2 + \alpha_0^3\beta_2\lambda_1 - \alpha_0^4\lambda_1\lambda_2$$

$$\gamma_{21} = -k_{11}k_{12} + \alpha_0\beta_2\beta_1 - \alpha_0^2\beta_2\lambda_1 - \alpha_0^2\beta_1\lambda_2 \\ + 2\alpha_0^3\lambda_2\lambda_1 - \alpha_2\alpha_1 + \alpha_0\alpha_2\beta_1 + \alpha_0\alpha_1\beta_2 - \alpha_0^2\alpha_2\lambda_1 \\ - \alpha_0^2\alpha_1\lambda_2 - \alpha_0^2\beta_2\beta_1 + \alpha_0^3\beta_2\lambda_1 + \alpha_0^3\beta_1\lambda_2 - \alpha_0^4\lambda_2\lambda_1$$

where the k_{ij} terms are the elements of the replacement matrix \mathbf{K} .

4. Empirical application

To provide an empirical application, we estimate the model using annual expenditure data on three expenditure categories, food (1), shelter (2), and transportation (3), from the Canadian *Survey of Household Spending*, for each province in Canada over the period from 1997 to 2009. The associated prices for these expenditure categories are normalized (with 2002=100) annual consumer price indices from Statistics Canada, CANSIM II Table 3260021. We use single member households who have positive spending on all three categories and end up with 12,153 observations. To estimate the share equation system (3), a stochastic version is specified and the third equation is dropped as this is a singular equation system; see Serletis and Shahmoradi (2007) for more details regarding estimation issues.

Table 1 contains a summary of results in terms of parameter estimates (with standard errors in parentheses), positivity, monotonicity, and curvature violations, as well as log likelihood values, when the model is estimated without the curvature conditions imposed (in the first column) and with the curvature conditions imposed (in the second column). Although the unrestricted model violates positivity and monotonicity at only three data points and curvature at only two data points when curvature conditions are not imposed (see the first column), the imposition of local curvature reduces the number of curvature violations (as well as those of positivity and monotonicity) to zero, thereby achieving theoretical regularity at every data point in the sample. Moreover, this is achieved without compromising much of the flexibility of the functional form.

5. Conclusion

Motivated by the widespread practice of ignoring the theoretical regularity conditions, Barnett (2002) has argued that flexible functional forms should be estimated subject to theoretical regularity, in order to produce evidence consistent with neoclassical microeconomic theory. In this note, we have illustrated a simple procedure for imposing curvature conditions locally in

the quadratic AIDS (QUAIDS) model of Banks et al. (1997), building on earlier work in this area by Ryan and Wales (1998), Moschini (1999) and Serletis and Shahmoradi (2007). The procedure is rooted in the work by Lau (1978), Gallant and Golub (1984) and Diewert and Wales (1987) and exploits the Slutsky matrix of the indirect utility function. Of course, as noted by Barnett (2002), theoretical regularity should not be treated as being equivalent to curvature alone; instead it includes positivity, monotonicity, and curvature.

Other methods of imposing curvature are discussed in Serletis and Feng (forthcoming). In particular, Serletis and Feng (forthcoming) provide a comparison among three different methods of imposing theoretical regularity on flexible functional forms—reparameterization using Cholesky factorization (as we do in this paper), constrained optimization, and Bayesian methodology. An alternative recent method of imposing regularity is that of Blundell et al. (2011), who propose reweighting observations to impose the Slutsky condition, based on the Hall and Huang (2001) nonparametric method.

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